TAYLOR POLYNOMIALS

1. Goals of these notes:

(i) Learn how to construct Taylor polynomials of any degree to approximate a given function near a given point.

(ii) Use Taylor polynomials to approximate function values, solutions to equations, etc.

(iii) Use Taylor polynomials to classify relative maxima and minima in cases where the second-derivative test fails.

2. Five basic facts about Taylor polynomials

(i) Taylor polynomials provide approximations to a function $f(x)$ near a particular value x_0 of x.

(ii) Taylor polynomials can be constructed of any order, limited by the number of derivatives f has at x_0.

(iii) The tangent-line approximation at x_0 is the Taylor polynomial of order 1.

(iv) The higher the order of the Taylor polynomial, the better the approximation. More precisely, let $T_n(x)$ be the Taylor polynomial of order n for $f(x)$ at x_0. Define

$$\epsilon_n(x) = f(x) - T_n(x);$$

in other words, $\epsilon_n(x)$ is the error in the approximation of f by T_n at the point x. Then

$$\lim_{x \to x_0} \frac{\epsilon_n(x)}{(x - x_0)^n} = 0.$$

(Note that the bigger n is, the faster $\epsilon(x)$ approaches zero.)

(v) The Taylor polynomial T_n of order n at the point x_0 is given by the formula:

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

3. Examples

As an example we consider the function $f(x) = x\sin(\pi x)$, and $x_0 = 1$. With $x_0 = 1$, it just happens that the graph of f crosses the x-axis at x_0.

The first five Taylor polynomials are:

$$\begin{aligned}
T_0(x) &= 0; \\
T_1(x) &= -\pi(x - 1); \\
T_2(x) &= -\pi(x - 1) - \pi(x - 1)^2; \\
T_3(x) &= -\pi(x - 1) - \pi(x - 1)^2 + \frac{\pi^3}{6}(x - 1)^3; \\
T_4(x) &= -\pi(x - 1) - \pi(x - 1)^2 + \frac{\pi^3}{6}(x - 1)^3 + \frac{\pi^3}{6}(x - 1)^4.
\end{aligned}$$

(The next section shows how to calculate these.) Note that the *degree* d of each T_n equals n. This is not true in every case; but it is always true that the $d \le n$. Note also that each polynomials equals the previous one plus one additional term in $(x-1)^n$. This is always the case, except that the coefficient of the additional term is sometimes zero.

Figure 1 shows the graphs of T_1, T_2, T_3, and T_4. You can see that the approximation improves as the order increases, especially in the neighborhood of $x = x_0$ (where the graph crosses the x-axis).

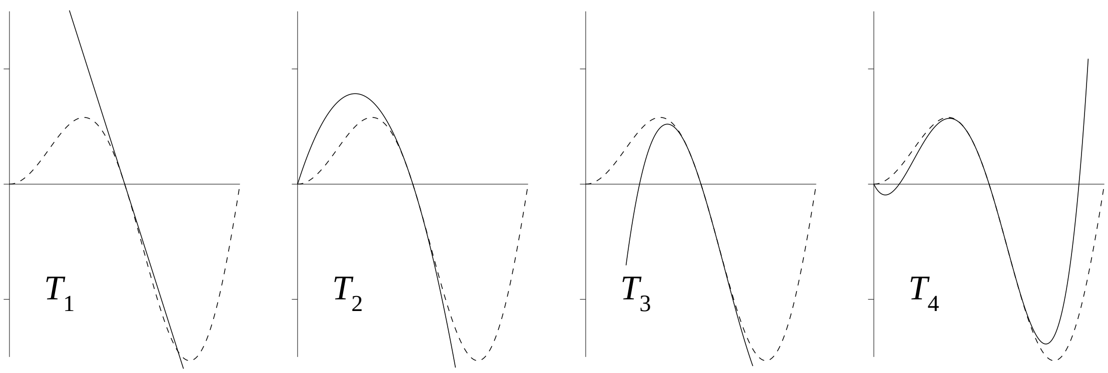

Figure 1

We can also see how the accuracy increases with the order by comparing values of $T_n(x)$ for a value of x near $x_0 = 1$ (such as $x = 1.3$) with the exact value of $f(1.3) = (1.3)\sin(1.3\pi) = -1.051722093$:

n	$T_n(1.3)$	error $= T_n(1.3) - f(1.3)$
0	0	1.051722093
1	−0.942477796	0.109244296
2	−1.225221135	−0.173499042
3	−1.085692890	−0.033979797
4	−1.043834416	0.007887676

The advantage of increasing n is greater for a value of x closer to x_0 (such as $x = 1.01$):

n	$T_n(1.1)$	error $= T_n(1.1) - f(1.1)$
0	0	1.051722093
1	−0.314159265	0.025759428
2	−0.345575191	−0.005656498
3	−0.340407479	−0.000488785
4	−0.339890707	0.000027985

4. Calculating Taylor polynomials

We recommend that you calculate Taylor polynomials in tabular form. As an example, we will calculate T_4 for $f(x) = x\sin(\pi x)$ around $x_0 = 1$. The headings of the table look like this:

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n

Step 1: Fill in the first column with values of n starting with 0 up to the order of the polynomial we want to calculate (4 in this case):

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n
0			
1			
2			
3			
4			

Step 2: Fill in the second column with the function $f(x)$, followed by its first, second, etc. derivatives:

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n
0	$x\sin(\pi x)$		
1	$\pi x\cos(\pi x) + \sin(\pi x)$		
2	$-\pi^2 x\sin(\pi x) + 2\pi\cos(\pi x)$		
3	$-\pi^3 x\cos(\pi x) - 3\pi^2\sin(\pi x)$		
4	$\pi^4 x\sin(\pi x) - 4\pi^3\cos(\pi x)$		

Step 3: Fill in the third column by plugging the value of x_0 (1 in this case) into the formulae in the second column. So the first value is $1 \cdot \sin(\pi \cdot 1) = 0$, the next value is $\pi \cdot 1 \cdot \cos(\pi \cdot 1) + \sin(\pi \cdot 1) = -\pi$, etc.:

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n
0	$x\sin(\pi x)$	0	
1	$\pi x\cos(\pi x) + \sin(\pi x)$	$-\pi$	
2	$-\pi^2 x\sin(\pi x) + 2\pi\cos(\pi x)$	-2π	
3	$-\pi^3 x\cos(\pi x) - 3\pi^2\sin(\pi x)$	π^3	
4	$\pi^4 x\sin(\pi x) - 4\pi^3\cos(\pi x)$	$4\pi^3$	

Step 4: Fill in the fourth column by dividing the values in the previous column by $n!$. (Recall that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, etc.; and that $0!$ is defined to be 1):

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n
0	$x\sin(\pi x)$	0	$0/0! = 0$
1	$\pi x\cos(\pi x) + \sin(\pi x)$	$-\pi$	$-\pi/1! = -\pi$
2	$-\pi^2 x\sin(\pi x) + 2\pi\cos(\pi x)$	-2π	$-2\pi/2! = -\pi$
3	$-\pi^3 x\cos(\pi x) - 3\pi^2\sin(\pi x)$	π^3	$\pi^3/3! = \pi^3/6$
4	$\pi^4 x\sin(\pi x) - 4\pi^3\cos(\pi x)$	$4\pi^3$	$4\pi^3/4! = \pi^3/6$

Step 5: Put the polynomial together by using the values in the fourth column as coefficients of powers of $(x - x_0)$ [in this case, $(x - 1)$]:

$$T_4(x) = 0 - \pi(x-1) - \pi(x-1)^2 + \frac{\pi^3}{6}(x-1)^3 + \frac{\pi^4}{6}(x-1)^4.$$

Note that we can get all Taylor polynomials of smaller orders by removing terms from T_4:

$$\begin{aligned}
T_3(x) &= 0 - \pi(x-1) - \pi(x-1)^2 + \frac{\pi^3}{6}(x-1)^3; \\
T_2(x) &= 0 - \pi(x-1) - \pi(x-1)^2; \\
T_1(x) &= 0 - \pi(x-1); \\
T_0(x) &= 0.
\end{aligned}$$

The foregoing calculation can be summarized by the formula:

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

This is the unique polynomial of degree not exceeding n which matches the values of f and all its derivatives up to order n at x_0 (the degree is less than n if $f^{(n)}(x_0)$ happens to be zero.

Exercise: Calculate T_6 for the function $f(x) = e^{3x}$ around $x_0 = 0$. Use the space below.

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$	c_n

Answer:

$$T_6(x) = 1 + 3(x-0) + \frac{9}{2}(x-0)^2 + \frac{9}{2}(x-0)^3 + \frac{27}{8}(x-0)^4 + \frac{81}{40}(x-0)^5 + \frac{81}{80}(x-0)^6.$$

5. Accuracy of Taylor polynomial approximation

Calculating Taylor polynomials of higher order entails more work. The advantage which offsets the greater work required is improved accuracy. This accuracy can be described as follows. The Taylor polynomial T_n of order n for a function f around a value x_0 can be calculated provided that f can be differentiated n times at x_0.

Let $\epsilon_n(x)$ be the error entailed in approximating f by T_n:

$$\epsilon_n(x) = f(x) - T_n(x).$$

Then

$$\lim_{x \to x_0} \frac{\epsilon_n(x)}{(x - x_0)^n} = 0.$$

(Note that the bigger n is, the faster $\epsilon(x)$ approaches zero.)

6. Exercises

Problems 1-10: Find the Taylor polynomials $T_0, \ldots, T_6$ for the given function f around the given value x_0.

1. $f(x) = e^x$, $x_0 = 0$
2. $f(x) = e^x$, $x_0 = 2$
3. $f(x) = \sin x$, $x_0 = 0$
4. $f(x) = \cos x$, $x_0 = 0$
5. $f(x) = \sin x$, $x_0 = \pi/2$
6. $f(x) = \ln x$, $x_0 = 1$
7. $f(x) = \sqrt{x}$, $x_0 = 1$
8. $f(x) = \sqrt{x}$, $x_0 = 9$
9. $f(x) = x \ln x$, $x_0 = 1$
10. $f(x) = x^4$, $x_0 = 2$

11. Sketch the graphs of the polynomials $T_0, \ldots, T_6$ from Problem 4 together with $f(x) = \cos x$ for $0 \leq x \leq 4\pi$.

12. (a) Use each of the polynomials $T_0, \ldots, T_6$ found in Problem 6 to estimate $\ln(1.1)$. Calculate the error in each approximation. (b) Repeat for $x = 1.001$. (c) Describe any patterns you see in the answers.

7. Application of Taylor polynomials: solving equations

Polynomial approximations similar to Taylor polynomials are a widely-used tool in science and engineering. The advantage of the approximation in some situations is simplicity: polynomials can be calculated using just the operations of addition and multiplication.

Example: Use Taylor polynomials to find approximate solutions to the equation

$$x \sin(\pi x) = 0.2.$$

We know no algebraic method for directly solving this equation. Taylor polynomial approximations let us use known techniques from algebra for solving polynomial equations.

Since $f(x) = x \sin(\pi x)$ is the same function used as an example in Sections 3 and 4, we have already calculated some Taylor polynomials for this function around the value $x_0 = 1$, in particular:

$$T_1(x) = -\pi(x-1);$$
$$T_2(x) = -\pi(x-1) - \pi(x-1)^2.$$

Note also that the base value $x_0 = 1$ solves the equation $f(x) = 0$, which is close to our equation $f(x) = 0.2$. Thus we can expect that our equation might have a solution near the value $x_0 = 1$, where the Taylor polynomials are a good approximation.

First we solve $T_1(x) = 0.2$. In other words,

$$-\pi(x-1) = 0.2$$

or $x = 1 - 0.2/\pi = 0.93634$. We will call this value x_1. How good is this solution to our equation? We can judge by comparing $f(x_1)$ with the desired value 0.2. In fact,

$$f(x_1) = x_1 \sin(\pi x_1) = 0.18602,$$

with an error of $0.18602 - 0.2 = -0.01398$.

(By the way, it may not be obvious, but this calculation is equivalent to one step of Newton's method, which you will study later.)

For a better approximation, we solve $T_2(x) = 0.2$. In other words,

$$-\pi(x-1) - \pi(x-1)^2 = 0.2.$$

We can use the quadratic formula to solve for $(x-1)$:

$$x - 1 = \frac{-\pi \pm \sqrt{\pi^2 - 4(0.2)(\pi)}}{2(\pi)} = -0.93167 \text{ or } -0.06833,$$

so $x = 0.06833$ (we call this value x_{2-}) or $x = 0.93167$ (we call this value x_{2+}).

We can check the quality of these approximations by calculating the value of f: $f(x_{2-}) = 0.01456$ (with an error of $0.01456 - 0.2 = -0.18544$ and $f(x_{2+}) = 0.19847$ (with an error of $0.19847 - 0.2 = 0.00153$). It is not surprising that x_{2+} works much better than x_{2-} since x_{2+} is closer to the base value x_0.

8. Application of Taylor polynomials: local extrema

Let's review the second-derivative test we learned previously for identifying local maxima or minima (rephrased):

(i) *If $f'(x_0)$ exists and is nonzero, then f does not have a local extremum at x_0.*

(ii) *If $f'(x_0) = 0$ and $f''(x_0)$ exists and is positive (negative) then f has a local minimum (maximum) at x_0.*

(iii) *If $f'(x_0) = 0$ and $f''(x_0) = 0$ then x_0 may be a local maximum or local minimum for f, or neither.*

In Case (iii) the criterion gives us no information about the behavior of f near x_0. We can use Taylor polynomials to resolve many instances of Case (iii).

Example 1: Consider the function $f(x) = -2\cos x - x^2$ near the point $x_0 = 0$. Taking derivatives, we find

$$f'(x) = 2\sin x - 2x, \ f'(x_0) = 0;$$
$$f''(x) = 2\cos x - 2, \ f''(x_0) = 0,$$

so this is a case where the second-derivative test above fails to yield an answer.

How can we use Taylor polynomials to resolve this situation? Consider the polynomial T_8 of order 8:

$$T_8(x) = -2 - \frac{(x-x_0)^4}{12} + \frac{(x-x_0)^6}{360} - \frac{(x-x_0)^8}{20160}.$$

Why should we use T_8 rather than some other order? In fact, T_8 is *not* the best choice for the following reason. In order to determine whether f has a local extremum at $x_0 = 0$, we are concerned only with values of x near x_0. When x gets close enough to x_0, $(x-x_0)^4/12$ will be much larger in size than both $(x-x_0)^6/360$ and $(x-x_0)^8/20160$. Thus the only really important terms in the polynomial above are the first two nonzero terms:

$$T_8(x) = -2 - \frac{(x-x_0)^4}{12} + \cdots = T_4(x) + \cdots.$$

The behavior of f near x_0 in this case matches that of T_4. Now we just have to recall that the graph of $-x^4$ has a local maximum at 0. Thus the graph of T_4 has a local maximum at x_0.

Example 2: Consider the function $f(x) = 64\sqrt{x} - 24x + x^2$ near the point $x_0 = 4$. Calculating derivatives, we find:

$$f'(x) = \frac{32}{\sqrt{x}} - 24 + 2x, \ f'(x_0) = \frac{32}{2} - 24 + 8 = 0;$$
$$f''(x) = -\frac{16}{x^{3/2}} + 2, \ f''(x_0) = -\frac{16}{8} + 2 = 0,$$

so the second-derivative test again fails to yield an answer. Taking one additional derivative, we find:

$$f'''(x) = \frac{24}{x^{5/2}}, \ f'''(x_0) = \frac{24}{32} = \frac{3}{4}.$$

Noting that $f(x_0) = 48$, we can calculate T_3:

$$T_3(x) = 48 + \frac{1}{8}(x - x_0)^3.$$

The behavior of f near x_0 matches that of T_3, and the behavior of T_3 near x_0 matches that of the function x^3 near 0. Since x^3 has no local extremum at 0, f therefore has no local extremum at x_0.

It is not necessarily actually to calculate a Taylor polynomial to decide whether f has a local extremum at a given point. We can streamline the calculation to yield the following criterion, which extends the second-derivative test studied earlier.

Evaluate successive derivatives of f at x_0 until a nonzero value is found.

(i) *If the order of the first nonzero derivative is odd, then f has no local extremum at x_0.* (By the way, this implies $f'(x_0) = 0$.)

(ii) *If the order of the first nonzero derivative is even, then f has a local extremum at x_0. This extremum is a minimum (maximum) if the value of the derivative is positive (negative).*

Example 3: Consider the function $f(x) = -2\cos x - x^2$ near the point $x_0 = 0$ (same as Example 1). Evaluate derivatives at $x_0 = 0$ until we get a nonzero value:

$$\begin{aligned} f'(x) &= 2\sin x - 2x, \;\; f'(x_0) = 0; \\ f''(x) &= 2\cos x - 2, \;\; f''(x_0) = 0; \\ f'''(x) &= -2\sin x, \;\; f'''(x_0) = 0; \\ f^{iv}(x) &= -2\cos x, \;\; f^{iv}(x_0) = -2. \end{aligned}$$

Since the fourth derivative is the first with a nonzero value, and is negative, we know that f behaves near x_0 like $-x^4$ behaves near 0, and has a local maximum.

Example 4: Consider the function $f(x) = 64\sqrt{x} - 24x + x^2$ near the point $x_0 = 4$ (same as Example 2). Evaluate derivatives at $x_0 = 0$ until we get a nonzero value:

$$\begin{aligned} f'(x) &= \frac{32}{\sqrt{x}} - 24 + 2x, \;\; f'(x_0) = \frac{32}{2} - 24 + 8 = 0; \\ f''(x) &= -\frac{16}{x^{3/2}} + 2, \;\; f''(x_0) = -\frac{16}{8} + 2 = 0; \\ f'''(x) &= \frac{24}{x^{5/2}}, \;\; f'''(x_0) = \frac{24}{32} = \frac{3}{4}. \end{aligned}$$

Since the third derivative is the first with a nonzero value, and is positive, we know that f behaves near x_0 like x^3 near 0, and thus has no local extremum.

In many cases this method will successfully classify a point as a relative maximum, relative minimum, or neither for a function. However, in some cases it still fails to yield an answer. One possibility is that all derivative values for a function may be zero at a point, even though the function is not constant. Such functions are not simple to describe, but one example is:

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

9. Exercises

Problems 1-4: (a) Find the Taylor polynomials for f of orders 1 and 2 about the given value x_0; (b) Use the polynomials from part (a) to find approximate solutions to the given equation; (c) Calculate the error in the resulting value of f for each approximate solution from part (b).

1. (a) $f(x) = x + \cos x$, $x_0 = 0$; (b) $x + \cos x = 1.15$
2. (a) $f(x) = e^x - x^2$, $x_0 = 0$, (b) $e^x - x^2 = .9$
3. (a) $f(x) = x^{1/3} + x^{1/5}$, $x_0 = 1$; (b) $x^{1/3} + x^{1/5} = 2.2$
4. (a) $f(x) = \ln x - x$, $x_0 = 1$; (b) $\ln x - x = -1.1$

Problems 5-12: Use the method of Section 8 to identify the given value x_0 as a local maximum, local minimum, or neither for the given function f.

5. $f(x) = 9x^5 - 20e^{3x-3} + 15x$, $x_0 = 1$
6. $f(x) = \ln x + e^{1-x/2}$, $x_0 = 2$
7. $f(x) = \tan x - 2x$, $x_0 = \pi/4$
8. $f(x) = \cos x + \sec x$, $x_0 = 0$
9. $f(x) = e^x - \ln(x+1) + 2\cos x$, $x_0 = 0$
10. $f(x) = \tan x - \sin x$, $x_0 = 0$
11. $f(x) = (x^3 - 6x^2 + 15x - 16)e^x$, $x_0 = 1$
12. $f(x) = (2 + x^2)\cos x$, $x_0 = 0$

13. Show that the method of Section 8 fails to classify the value $x_0 = 0$ for the function

$$f(x) = x^{10/3}.$$

Explain why. Classify this point by graphing the function.

14. Repeat Problem 13 using the function

$$f(x) = x^{11/3}.$$

Answers to exercises

From Section 6:

In the following, T_0, T_1, ..., T_5 can be obtained by truncating T_6.

1. $T_6(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}$.
2. $T_6(x) = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{(x-2)^5}{120} + \frac{(x-2)^6}{720}$.
3. $T_6(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$.
4. $T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$.
5. $T_6(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24} - \frac{(x-\pi/2)^6}{720}$.
6. $T_6(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$.
7. $T_6(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024}$.
8. $T_6(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \frac{5}{279936}(x-9)^4 + \frac{7}{3038848}(x-9)^5 - \frac{7}{60466176}(x-9)^6$.
9. $T_6(x) = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{20} + \frac{(x-1)^6}{30}$.
10. $T_6(x) = 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4$.

From Section 9:

In the following, x_{2+} and x_{2-} may be interchanged, depending on the form into which the quadratic equation is put before solving. The important thing is to choose the one of x_{2+} and x_{2-} which is closer to x_0.

1. (a) $x_1 = .15$, $x_{2-} = 0.16334$. (b) $f(x_1) = 1.13877$ (error of -0.01123), $f(x_{2-}) = 1.15003$ (error of 0.00003).

2. (a) $x_1 = -.1$, $x_{2-} = -0.09545$. (b) $f(x_1) = 0.89484$ (error of -0.00516), $f(x_{2-}) = 0.89986$ (error of -0.00014).

3. (a) $x_1 = 1.375$, $x_{2-} = 1.44641$. (b) $f(x_1) = 2.17775$ (error of $-.02225$), $f(x_{2-}) = 2.20753$ (error of 0.00753).

4. (a) x_1 does not exist, $x_{2+} = 1.44721$, $x_{2-} = 0.55279$ (equidistant from x_0). (b) $f(x_{2+}) = -1.07757$ (error of 0.02243), $f(x_{2-}) = -1.14557$ (error of $-.04557$).

5. maximum 6. neither 7. minimum 8. minimum 9. neither 10. neither 11. minimum 12. maximum