# GEOMETRIC SERIES

## 1. Goals of these notes:

(i) Learn to recognize a geometric series.

(ii) Be able to find the sum of a finite geometric series.

(iii) Use geometric series to solve saving problems.

(iv) Understand convergent and divergent infinite series.

(v) Be able to determine whether an infinite geometric series is convergent, and find the sum if it exists.

## 2. Definition and classification of geometric series

A *series* is simply an ordered set of terms to be added. The following are examples of series:
(a) $1 + 2 + 3 + 4 + 5$
(b) $3 + 3 + 1 + 4 + 10 + 3 + 2$
(c) $9 - 9 + 9 - 9$
(d) $1 + 2 + 4 + 8 + \cdots + 2^{120}$
(e) $1 + 3 + 5 + 7 + \cdots$
(f) $10 + \frac{10}{3} + \frac{10}{9} + \cdots \frac{10}{3^{90}}$
(g) $1 - 10 + 100 - 1000 + \cdots$

From these examples, we see that the terms of the series may or may not follow an obvious pattern, and also may be finite or infinite in number.

Here we are mainly concerned with a particular types of series called *geometric*. A geometric series is one whose terms are created according to the pattern:

$$a + ar + ar^2 + ar^3 + \cdots.$$

The terms of a geometric series may be finite or infinite in number.

Another way to recognize a geometric series is that *the ratio of any two consecutive terms is always the same value*. For example, consider example (a) above:

$$1 + 2 + 3 + 4 + 5.$$

Taking the ratio of the second to the first term, we get $2/1$. Taking the ratio of the third to the second term, we get $3/2$. Since $2/1 \neq 3/2$, this is not a geometric series.

On the other hand, consider example (c) above:

$$9 - 9 + 9 - 9.$$

Taking the ratio of the second to the first term, we get $-9/9 = -1$. Taking the ratio of the third to the second term, we get $9/-9 = -1$. Taking the ratio of the fourth to the third term, we get $-9/9 = -1$. Since every one of these yields the same value $-1$, this is a geometric series.

*Question:* Which of the other examples above are geometric series? *Answer:*

Geometric series are not only a useful in their own right, but also serve as an important benchmark for comparison with other, more complicated types of series. We will use them as such when we study Taylor series.

Any geometric series can be described in terms of just three numbers: $a$ (the first term), $r$, the ratio between two consecutive terms, and $n$ (the number of terms). For example, in example (c) above, $a$ equals 9 (the value of the first term). $r$ equals $-1$ (the ratio of any two consecutive terms), and $n$ equals 4 (the number of terms). In terms of these three values, the series could be written in the form:

$$a + ar + \cdots + ar^{n-1},$$

or, in this case:

$$9 + 9(-1) + 9(-1)^2 + 9(-1)^3.$$

## 3. Summing finite geometric series

The sum of a finite geometric series can be calculated by directly adding the terms. For example:

$$2 + 6 + 18 + 54 + 162 = 242.$$

However, if the series contains many more terms:

$$2 + 6 + 18 + 54 + 162 + \cdots + 2 \cdot 3^{95},$$

we might want a faster method. Owing to the special form of a geometric series, there is a gimmick which lets us calculate the sum with the same amount of calculation, no matter how many terms are involved.

The first step is to determine the value of $r$ by dividing two consecutive terms. (It's also a good idea to make sure at this point that the series is indeed a geometric series.) In this case

$$r = 6/2 = 3.$$

Now we call the value of the sum $S$:

$$S = 2 + 6 + 18 + 54 + 162 + \cdots + 2 \cdot 3^{95}.$$

Directly below this equation, we rewrite the same equation with both sides multiplied by $r$ (3 in this case):

$$\begin{array}{rcccccccccccccc} S & = & 2 & + & 6 & + & 54 & + & 162 & + & \cdots & + & 2 \cdot 3^{95} & & \\ 3S & = & & & 6 & + & 54 & + & 162 & + & \cdots & + & 2 \cdot 3^{95} & + & 2 \cdot 3^{96} \end{array}$$

Due to the special nature of the geometric series (the ratio of two consecutive terms yielding always the same value) this multiplication converts each term of the series into the next one—2 becomes 6, 6 becomes 18, and so on. We write the second line shifted one term to the right to reflect this. The last term $2 \cdot 3^{95}$ of the series becomes $2 \cdot 3^{96}$, which is written way out to the right of the original series.

Then subtract the two equations:

$$\begin{array}{rrcccccccccccccc}
 & S & = & 2 & + & 6 & + & 54 & + & 162 & + & \cdots & + & 2\cdot 3^{95} & & \\
- & 3S & = & & & 6 & + & 54 & + & 162 & + & \cdots & + & 2\cdot 3^{95} & + & 2\cdot 3^{96} \\
\hline
 & -2S & = & 2 & + & 0 & + & 0 & + & 0 & + & \cdots & + & 0 & - & 2\cdot 3^{96}
\end{array}$$

The magic of this gimmick is that all the terms cancel out except the first and last. We can solve the resulting equation for $S$:

$$-2S = 2 - 2\cdot 3^{96};$$

$$S = \frac{2 - 2\cdot 3^{96}}{-2}.$$

*Exercise:* Use this method to find the sum of the geometric series:

$$3 + 6 + 12 + 24 + 48 + 96.$$

Compare the result with direct addition.

*Answer:*

$$S = \frac{3 - 192}{-1}.$$

Direct addition gives the value 189, which is equal to this.

This gimmick can be used to find the sum of any geometric series, except in one special circumstance. What is it?

## 4. Application: Financial problems

Suppose Alfonso opens a savings account at age 20 with a \$1000 deposit. Each year thereafter, he deposits another \$1000, for a total of \$40,000. The savings account earns 4% interest, compounded annually. How much money has he saved at the time of the fortieth deposit?

An important principle to keep in mind when doing this and similar problems is that "adding" interest is actually a matter of *multiplication*. To add 4% interest to a quantity, we multiply by 1.04. In general, to add $p$% interest to a quantity, we multiply that quantity by $(1 + p/100)$.

After the first deposit, Alfonso has just \$1000. After the second deposit, he has the original \$1000 (with interest) and an additional \$1000:

$$A_2 = 1000 + 1.04(1000).$$

After the third deposit, Alfonso has the quantity $A_2$ (with interest) and an additional \$1000:

$$\begin{aligned} A_3 &= 1000 + 1.04A_2 = 1000 + 1.04(1000 + 1.04[1000]) \\ &= 1000 + 1.04(1000) + 1.04^2(1000). \end{aligned}$$

After the fourth deposit, Alfonso has the quantity $A_3$ (with interest) and an additional \$1000:

$$\begin{aligned} A_4 &= 1000 + 1.04A_3 = 1000 + 1.04(1000 + 1.04[1000] + 1.04^2[1000]) \\ &= 1000 + 1.04(1000) + 1.04^2(1000) + 1.04^3(1000). \end{aligned}$$

Do you see the pattern? After the $n$th deposit, Alfonso has:

$$A_n = 1000 + 1.04(1000) + 1.04^2(1000) + \cdots + 1.04^{n-1}(1000).$$

In particular,

$$A_{40} = 1000 + 1.04(1000) + 1.04^2(1000) + \cdots + 1.04^{39}(1000).$$

And this is a geometric series. We can use the gimmick from the previous section to find the sum:

$$A_{40} = \frac{1000 - 1.04^{40}(1000)}{1 - 1.04} = 95025.50,$$

or, in other words, \$95,025.50.

Now you try this problem: Deirdre opens a savings account at age 20 at 6% interest and deposits \$2500 each year. How much money has she saved after making 45 deposits?

*Answer:* \$531,859.

**5. Infinite series**

Recall that the terms of a series may be finite or infinite in number. Up to this point we have focused on finite series. Infinite series must be handled cautiously because your intuition for dealing with finite sums does not necessarily extend to infinite sums. For example, a finite sum takes the same value no matter what the order of the terms, but this is true for some infinite sums and not others.

Consider for example, this infinite series:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

What is the sum?

We could group the terms in pairs, like so:

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots,$$

which makes it appear that the sum should be zero. On the other hand, we could group all terms but the first one in pairs, like so:

$$1 + (-1 + 1) + (-1 + 1) + \cdots,$$

which makes it appear that the sum should be 1. So is the sum 0, 1, or something else?

The moral of this little example is that infinite series are treacherous and complicated and one must exercise extreme caution in their presence. The sum of an ininite series is defined in a very specific way to avoid contradictions such as we saw above. We approach the sum of an infinite series by way of the sums of finite series (which are easy to work with). For example, consider the series:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots.$$

To find the sum, we add one term at a time:

$$\begin{aligned}
&\frac{1}{1\cdot 2}=\frac{1}{2}\\
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{2}{3}\\
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{3}{4}\\
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=\frac{4}{5}\\
&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}=\frac{5}{6}\\
&\cdots
\end{aligned}$$

These are called *partial sums* of the series. So far, there appears to be a pattern that the partial sum of the first $n$ terms takes the value

$$\frac{n}{n+1}.$$

Assuming we could prove that this formula holds for all $n$, then the sum of the series is defined to be the *limit* of the partial sums:

$$\lim_{n\to\infty}\frac{n}{n+1}=1.$$

If the limit exists, we say the sum *converges* and its value is the value of the limit. If the limit does not exist (which is not unusual), then we say the sum *diverges* and it has no value.

This applies to all infinite series. We restate it once more:

*Let*

$$a_1+a_2+a_3+a_4+\cdots$$

*be an infinite series. Define $S_n$ to be the $n$th partial sum:*

$$S_n=a_1+a_2+\cdots+a_n.$$

*If*

$$\lim_{n\to\infty}S_n$$

*exists, then the series is said to* converge *and its sum is defined to be the value of the limit. If the limit does not exist then the series is said to* diverge *and its sum is undefined.*

Let's apply this criterion to the example we started this section with:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

Calculate the first few partial sums:

$$\begin{aligned}
&1 = \\
&1 - 1 = \\
&1 - 1 + 1 = \\
&1 - 1 + 1 - 1 = \\
&1 - 1 + 1 - 1 + 1 = \\
&1 - 1 + 1 - 1 + 1 - 1 =
\end{aligned}$$

Do you see the pattern? What is the limit of the sequence of partial sums? And therefore what is the sum of the series?

In practice the difficulty in applying this criterion is finding a formula for the partial $S_n$. In most cases this is difficult or impossible. All kinds of complicated indirect methods have been devised to find the sum of particular series or even just to determine whether the series converges.

The sum of an infinite geometric series, however, is always determinable because we have a nice gimmick for adding up a finite geometric series. Any infinite geometric series can be put in the form:

$$a + ar + ar^2 + ar^3 + \cdots.$$

A partial sum of this series takes the form:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Using the gimmick of Section 3, we can find this to be:

$$S_n = \frac{a - ar^n}{1 - r}.$$

Now the question is to calculate:

$$\lim_{n\to\infty} \frac{a - ar^n}{1 - r}.$$

This depends on the value of $r$. If, for example, $r = 1.1$, then $r^n$ becomes larger and larger as $n$ increases and the limit does not exist. If $r = 0.9$, then $r^n$ goes to zero as $n$ increases to $\infty$. The limit turns out to be:

$$\lim_{n\to\infty} = \begin{cases} \frac{a}{1-r}, & \text{if } |r| < 1\ ; \\ \text{undefined} & \text{otherwise} \end{cases}$$

and this is the formula for the sum of an infinite geometric series.

*Example:* Find

$$\frac{3}{5} - \frac{3}{25} + \frac{3}{125} - \frac{3}{625} + \cdots.$$

We first calculate $a$ and $r$. The value of the first term gives us $a = 3/5$. The ratio of the second and first terms gives us $r = -1/5$. Since $|-1/5| < 1$, the series converges. The sum is given by

$$\frac{a}{1-r} = \frac{3/5}{1+1/5} = \frac{1}{2}.$$

*Example:* Find

$$5 + 20 + 80 + 320 + 1280 + \cdots.$$

We first calculate $a$ and $r$. The value of the first term gives us $a = 5$. The ratio of the second and first terms gives us $r = 4$. Since $|4| > 1$, the series diverges. Thus the sum is undefined.

## 6. Exercises

Problems 1-8: Determine whether the given series is a geometric series. If it is, find the values of $a$, $r$, and $n$.

1. $1 + 4 + 9 + 16 + \cdots + 400$.
2. $1 - 3 + 5 - 7 + \cdots$.
3. $60 + 6 + 0.6 + .06 + .006$.
4. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{500}$.
5. $4 - 12 + 36 - 108 + 324$.
6. $7 + \frac{7}{2} + \frac{7}{4} + \cdots + \frac{7}{2^{150}}$.
7. $3 - 3 + 3 - 3 + 3 - 3 + \cdots$.
8. $400 + 4000 + 40000 + \cdots + 4 \cdot 10^{133}$.

Problems 9-12: Find the sum of the geometric series.

9. $4 - 12 + 36 - 108 + 324 + \cdots + 4 \cdot 3^{400}$.
10. $7 + \frac{7}{2} + \frac{7}{4} + \cdots + \frac{7}{2^{150}}$.
11. $400 + 4000 + 40000 + \cdots + 4 \cdot 10^{133}$.
12. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{35}}$.

13. In what circumstance does the method of Section 3 fail to give an answer? How can you find the sum of the series in this case?

14. Claudine opens a savings account at age 20 with a \$2000 deposit. The account earns 8% interest, compounded yearly, and she continues to make \$2000 deposits every year for a total of 30 deposits. How much money is in the account after she makes the 30th deposit?

15. George opens a savings account at age 20 with a \$2000 deposit. The account earns 1% interest, compounded yearly, and he continues to make \$2000 deposits every year for a total of 60 deposits. How much money is in the account after he makes the 60th deposit?

16. Prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

*Hint:* What is

$$\frac{1}{n} - \frac{1}{n+1}?$$

Problems 17-22: Tell whether the given infinite series is geometric. If it is geometric, determine whether it converges. If it converges, find the sum.

17. $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$.
18. $\frac{3}{11} + \frac{3^2}{11} + \frac{3^3}{11} + \cdots$.
19. $\frac{6}{7} + \frac{6}{7^2} + \frac{6}{7^3} + \cdots$.
20. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$.
21. $5 - 5 + 5 - 5 + \cdots$
22. $8 + \frac{8}{9} + \frac{8}{81} + \frac{8}{729} + \cdots$.

**Answers to exercises**

1. no.
2. no.
3. yes; $a = 60$, $r = 0.1$, $n = 5$.
4. no.
5. yes; $a = 4$, $r = -3$, $n = 5$.
6. yes; $a = 7$, $r = 1/2$, $n = 151$.
7. yes; $a = 3$, $r = -1$, $n = \infty$.
8. yes; $a = 400$, $r = 10$, $n = 132$.
9. $1 - (-3)^{401}$.
10. $2(7 - 7 \cdot 2^{-151})$.
11. $(400/9)(10^{132} - 1)$.
12. $2(1 - 1/2^{36})$.
14. \$226,566.42.
15. \$163,339.34.
17. geometric; sum= 1/6.
18. geometric; divergent.
19. geometric; sum=1.
20. not geometric.
21. geometric; divergent.
21. geometric; sum=9.