# SM261 FINAL EXAMINATION
## 14 DECEMBER 2006

PART ONE: NO CALCULATORS
When you are finished with PART ONE hand it in and begin work on PART TWO.

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1. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{bmatrix}$.

a. Calculate $AB$.
b. Calculate $B^T A^T$.

2. Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix}$. Find $C^{-1}$.

3. Find all solutions to the following system of equations. Write your solutions in vector form.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 2 \\ 2x_1 & + & 2x_2 & - & x_3 & - & x_4 & + & x_5 & = & -1 \\ 4x_1 & + & 4x_2 & - & 3x_3 & - & x_4 & + & 3x_5 & = & 3 \end{array}$$

4. Identify the redundant vectors among the list of vectors below.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}.$$

5. Use row reduction techniques to find det($A$) if $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

6. Let $T$ be the linear transformation determined by $T(e_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

a. Find the matrix of $T$ with respect to the standard basis $\{e_1, e_2\}$.

b. Find the matrix of $T$ with respect to the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

c. Is $T$ an orthogonal linear transformation? Explain.

7. Let $A$ be the matrix $\begin{bmatrix} 16 & 9 \\ -4 & 4 \end{bmatrix}$.

a. Find all of the eigenvalues of the matrix $A$.

b. For one of the eigenvalues of the matrix $A$ compute the corresponding eigenspace.

8. Use Cramer's Rule to find the solutions to the system

$$\begin{array}{rcrcl} 2x & + & y & = & 4 \\ 3x & + & 10y & = & 3. \end{array}$$

Show your work.

END OF PART ONE

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PART TWO: CALCULATORS ARE PERMITTED

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 5 \\ 2 & 4 & 2 & 6 \\ 1 & 2 & 2 & 4 \end{bmatrix}$.

a. Find a basis for im($A$).

b. Find a basis for ker($A$).

2. Suppose $v_1$, $v_2$, and $v_3$ are non-zero vectors in $R^3$ that are orthogonal to each other, i.e. $0 = v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3$.
a. Explain why the three vectors are linearly independent.
b. Explain, using a), why the three vectors form a basis for $R^3$.

3. I have *17* bills in my pocket (*1*'s, *5*'s, and *10*'s) totalling *\$77*. How many of each type of bill do I have? (Use techniques from this course to solve.)

4. Let $A$ be a $10 \times 10$ invertible matrix. Explain your answers to the following.
a. What does it mean for $A$ to be invertible?
b. What are the possible values of the rank of $A$?
c. What are the possible values of the nullity of $A$?
d. What are the possible values of det($A$)?
e. Explain why for any $10 \times 1$ vector $b$ the equation $Ax = b$ is consistent, i.e. has a solution.

5. Suppose an $n \times n$ matrix $A$ satisfies the matrix equation $A^2 + 2A = I$, where $I$ is the $n \times n$ identity matrix. Show that $A$ is invertible.

6. Suppose $A$ is a $3 \times 8$ matrix.
a. What are the possible values of the rank of $A$?
b. What are the possible values of the nullity of $A$?
c. What are the possible values of the sum of the rank and nullity of $A$?

7. a. Given a subspace $V$ of $R^n$, define $V^\perp$ and explain why it is a subspace (of $R^n$).

b. Let $V$ be the subspace of $R^3$ with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Find a basis for $V^\perp$.

8. Let $V$ be the subspace of $R^4$ spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

a. Use the Gram-Schmidt method to find an orthonormal basis for $V$.

b. Find $proj_V \left( \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$, the projection of the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto $V$.

9. Suppose $v_1$, $v_2$, $v_3$, and $v_4$ are the *rows* of a $4\times 4$ matrix *A*, i.e., $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$.

Suppose also that det(*A*) = 2. Find the determinants of the following matrices. Explain your answers.

a. $\begin{bmatrix} v_3 \\ v_2 \\ v_1 \\ v_4 \end{bmatrix}$ b. $\begin{bmatrix} v_1 + 3v_2 \\ v_2 \\ 4v_3 \\ v_4 \end{bmatrix}$ c. $\begin{bmatrix} v_1 \\ v_2 \\ v_2 \\ v_4 \end{bmatrix}$

10. A matrix *A* has eigenvalues *2* and *3*.
a. Show that if $v$ is an eigenvector of *A* then it is also an eigenvector of $A^2$. What are the eigenvalues of $A^2$?
b. Show that if $v$ is an eigenvector of *A* then it is also an eigenvector of $A^{-1}$. What are the eigenvalues of $A^{-1}$?

11. Find the best (least squares) fit $y = c_0 + c_1 t$ to the data $(t, y) = (1,-1)$, (2,1), and (3,4).

12. Let $T$ be the linear transformation from $R^2$ to $R^2$ which is the projection onto the line $y = x$. Let *A* be the matrix of the linear transformation *T*.
a. Find *A*.
b. Find the eigenvalues and eigenvectors of the matrix *A*.
c. Use b) to find an invertible matrix *S* and a diagonal matrix *D* so that $S^{-1}AS = D$..

END OF PART TWO