Example of when CTFT doesn't exist, but Laplace transform does

$$x(t) = e^t u(t)$$

Find the CTFT of *x*(*t*):

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{0}^{\infty} e^t e^{-j2\pi ft} dt$$

$$= \int_{0}^{\infty} e^{(1-j2\pi f)t} dt = \frac{1}{1-j2\pi f}\left[e^{(1-j2\pi f)t} \Big|_{t=0}^{t=\infty} \right]$$

$$= \frac{1}{1-j2\pi f} e^t e^{-j2\pi ft} \Big|_{t=0}^{t=\infty}$$

$$= \frac{1}{1-j2\pi f}\left[\left(e^{\infty} e^{j2\pi f\infty} \right) - e^0 \right] = \infty$$

Find the Laplace Transform of *x*(*t*):

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{0}^{\infty} e^t e^{-st} dt$$

$$= \int_{0}^{\infty} e^{(1-s)t} dt = \frac{1}{1-s}\left[e^{(1-s)t} \Big|_{t=0}^{t=\infty} \right]$$

$$= \frac{1}{1-s} e^t e^{-(\sigma+j\omega)t} \Big|_{t=0}^{t=\infty} = \frac{1}{1-s} e^t e^{-\sigma t} e^{-j\omega t} \Big|_{t=0}^{t=\infty}$$

$$= \frac{1}{1-s} e^{(1-\sigma)t} e^{-j\omega t} \Big|_{t=0}^{t=\infty}$$

Note: As $t \to \infty$, $e^{(1-\sigma)t}$ will $\to$ 0 $\boxed{\text{IF}}$ 1-σ is less than 0

or in other words, $\sigma > 1$, which is also written as $\text{Re}\{s\} > 1$

So if we assume $\sigma > 1$, then

$$X(s) = \frac{1}{1-s} e^{(1-\sigma)t} e^{-j\omega t} \Big|_{t=0}^{t=\infty} = \frac{1}{1-s}\left[e^{-\infty} - e^0 \right] = \frac{1}{s-1}, \quad \text{Re}\{s\} > 1$$

Since the Laplace transform is not ∞, we say that the Laplace transform converges for any value of $s = \sigma + j\omega$, as long as Re{s} > 1. That is, the "Region of Convergence" (ROC) is Re{s} > 1.

With Laplace transforms, you may be asked to plot the ROC. This is done on a plot like that in Figure 1.

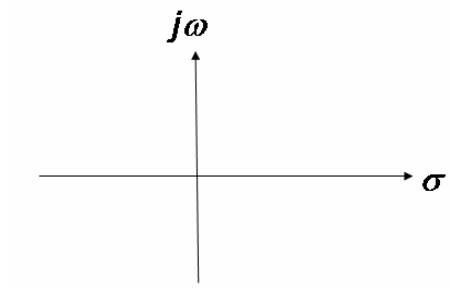


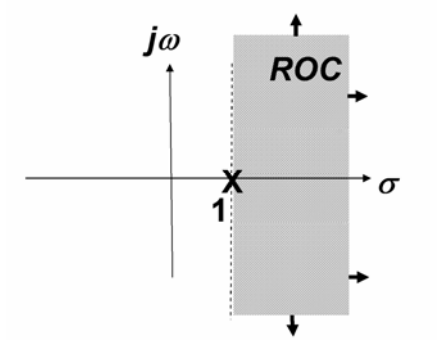
Figure 1: s-plane

On such a plot you should place the poles and zeros of X(s), and shade in the ROC. Mark the poles with an X and zeros with a circle.

-A pole of X(s) is a value of s that makes the ***denominator*** of X(s) = 0.

-A zero of X(s) is a value of s that makes the numerator of X(s) = 0.

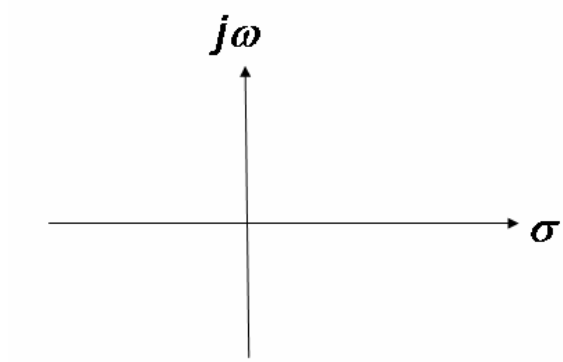
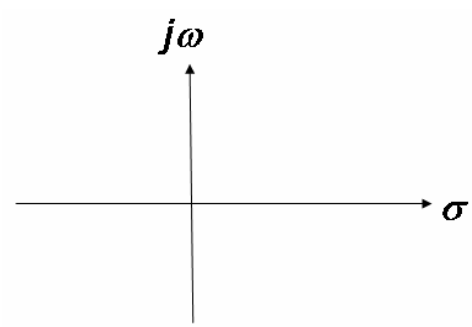
In this example, there are no zeros, but there is one pole at $s = \sigma + j\omega = 1$ (that is, $\sigma = 1$) and the ROC is $\mathrm{Re}\{s\} > 1$ (that is, $\sigma > 1$) . The plot of the ROC would look like:



Examples:

Plot the ROCs for:

1. $X(s) = \dfrac{s-1}{s^2-9}$,
$-3 < \mathrm{Re}\{s\} < 3$ (that is, $-3 < \sigma < 3$)



2. $X(s) = \dfrac{s-3}{(s-2+j4\pi)(s-2-j4\pi)}$,
$\mathrm{Re}\{s\} < 2$ (that is, $\sigma < 2$)