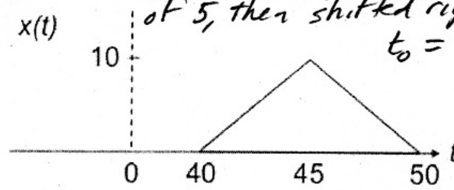
Name: Key

EE322 Fall 2012 Lab 07: Fourier Worksheet
(use the Continuous-time Fourier Transform Tables/Properties)

1. Evaluate the CTFT of $x(t)$ in the figure to the right:
Hint: write the steps needed to transform the $tri(t)$ signal to this, and apply the Fourier transform properties at each step.

a triangle time expanded by a factor of 5, then shifted right by $t_0 = 45$



$$10\,tri\left(\frac{t-45}{5}\right)$$

$$10\,tri(t) \longleftrightarrow 10\,sinc^2(f)$$

$$10\,tri\left(\frac{t}{5}\right) \longleftrightarrow 10 \cdot \frac{1}{|1/5|}\,sinc^2(5f) = 50\,sinc^2(5f)$$

$$10\,tri\left(\frac{t-45}{5}\right) \longleftrightarrow 50\,sinc^2(5f)\,e^{-j2\pi f(45)}$$

$$= \boxed{50\,sinc^2(5f)\,e^{-j90\pi f}}$$

indicated $\frac{1}{2}\frac{d}{dt}$

indicates time delay by $t_0 = 2$

2. Find the inverse CTFT (i.e., the time signal $x(t)$) corresponding to: $X(f) = j\pi f \dfrac{18}{16+4\pi^2 f^2} e^{-j4\pi f}$.

Hint: Start with a signal from the Fourier transform table that has a transform that looks like this, then determine which properties have been applied to the transform.

use: $$e^{-a|t|} \longleftrightarrow \frac{2a}{(2\pi f)^2 + a^2}$$

rewrite $X(f)$: $$X(f) = j2\pi f \cdot \left(\frac{1}{2}\;\frac{9}{4}\;\frac{2\cdot 4}{(2\pi f)^2 + 4^2}\right) e^{-j2\pi f(2)}$$

$$= \frac{9}{8}\; j2\pi f \left[\frac{2\cdot 4}{(2\pi f)^2 + 4^2}\right] e^{-j2\pi f(2)}$$

$$\longleftrightarrow \boxed{\frac{9}{8}\frac{d}{dt}\left[e^{-4|t-2|}\right]}$$

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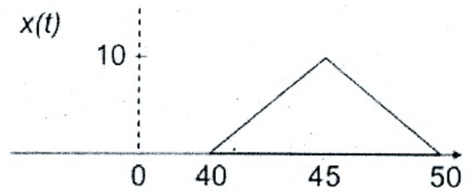
1. Evaluate the CTFT of $x(t)$ in the figure to the right:
Hint: write the steps needed to transform the *tri*(t) signal to this, and apply the Fourier transform properties at each step.



* IF you worked problem 2 the way it was originally written, The solution goes like this:

indicates $\frac{1}{2}\frac{d}{dt}$

indicates time delay by $t_0 = 2$

2. Find the inverse CTFT (i.e., the time signal $x(t)$) corresponding to: $X(f) = j\pi f \dfrac{18}{16+9\pi^2 f^2} e^{-j4\pi f}$

Hint: Start with a signal from the Fourier transform table that has a transform that looks like this, then determine which properties have been applied to the transform.

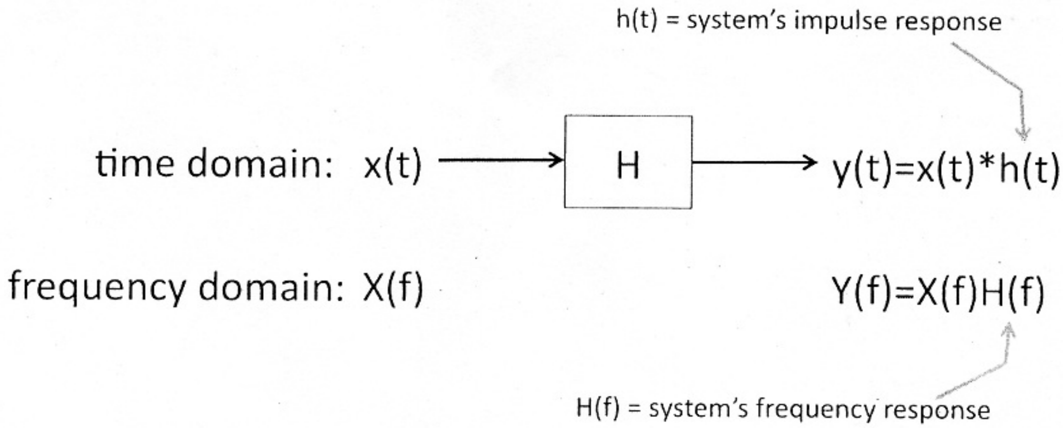
rewrite $X(f)$ as $X(f) = j2\pi f \ \frac{1}{2} \ \dfrac{18}{\frac{9}{4}\left(\frac{64}{9} + 4\pi^2 f^2\right)} e^{-j2\pi f(2)}$

$$X(f) = j2\pi f \ \frac{1}{2} \ \frac{\frac{27}{8}\left(2\cdot\frac{8}{3}\right)}{\frac{9}{4}\left(\frac{8}{3}^2+(2\pi f)^2\right)} e^{-j2\pi f(2)}$$

$$= j2\pi f\left(\frac{1}{2}\,\frac{27}{8}\,\frac{4}{9}\right)\left(\frac{\left(2\cdot\frac{8}{3}\right)}{\frac{8}{3}^2+(2\pi f)^2}\right) e^{-j2\pi f(2)}$$

$$= j2\pi f\left(\frac{3}{4}\right)\frac{\left(2\cdot\frac{8}{3}\right)}{\left(\frac{8}{3}\right)^2+(2\pi f)^2} e^{-j2\pi f(2)} \longleftrightarrow \boxed{\frac{3}{4}\frac{d}{dt}\left[e^{-\frac{8}{3}|t-2|}\right]}$$

Note: here is the relationship between a system's impulse response and its frequency response:



3. The *frequency response* ($H(f)$) of a system is equal to the Fourier transform of its *impulse respons* ($h(t)$). Determine the frequency response AND the impulse response to a system if the input is given as $x(t) = e^{-2t}u(t)$ and the output is given by $y(t) = 2e^{-2(t-3)}u(t-3)$. This is an example of ho the CTFT is commonly used to solve system problems. Work this problem in the following steps:

(a) Find the CTFT of the input, $x(t) = e^{-2t}u(t)$ (this is $X(f)$).

From table, $X(f) = \boxed{\dfrac{1}{j2\pi f + 2}}$

(b) Find the CTFT of the output, $y(t) = 2e^{-2(t-3)}u(t-3)$ (this is $Y(f)$).

using time shift property

$$Y(f) = \frac{2}{j2\pi f + 2} \cdot e^{-j2\pi f \cdot 3} = \boxed{\frac{2e^{-j6\pi f}}{j2\pi f + 2}}$$

(c) Now since $y(t) = x(t) * h(t)$, we can use the multiplication-convolution duality property which says that $Y(f) = X(f)H(f)$. This means that $H(f) = \frac{Y(f)}{X(f)}$. Find $H(f)$. This is the system's *frequency response*.

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\frac{2e^{-j6\pi f}}{\cancel{j2\pi f + 2}}}{\frac{1}{\cancel{j2\pi f + 2}}} \quad \boxed{= 2e^{-j6\pi f}}$$

rewrite as $\quad H(f) = 1 \cdot e^{-j2\pi f(3)}$

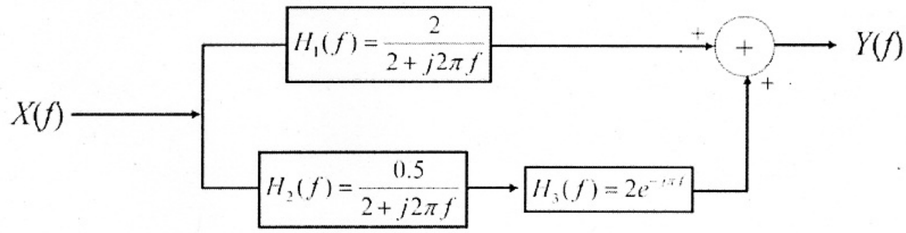
(d) Find the inverse CTFT of $H(f)$ to give $h(t)$, the system impulse response.

$$\delta(t) \longleftrightarrow 1$$

$$\delta(t-3) \longleftrightarrow 1 \cdot e^{-j2\pi f(3)} = e^{-j6\pi f}$$

so $\boxed{h(t) = 2\,\delta(t-3)}$

4. Evaluate the overall system frequency response ($H(f)$) and the overall system impulse response ($h(t)$) of the system shown to the right. Simplify your answer.



cascade

$$H_2(f)H_3(f) = \frac{e^{-j\pi f}}{2+j2\pi f}$$

rewrite as: $\frac{1}{2}\left(\frac{2e^{-j2\pi f\frac{1}{2}}}{2+j2\pi f}\right)$

$$H(f) = H_1(f) + H_2(f)\,H_3(f)$$

$$= \frac{2}{2+j2\pi f} + \frac{1}{2}\left(\frac{2}{2+j2\pi f}\right)e^{-j2\pi f\frac{1}{2}}$$

this indicates a delay of 1/2

then

$$\boxed{h(t) = 2e^{-2t}u(t) + e^{-2(t-\frac{1}{2})}u(t-\tfrac{1}{2})}$$

5. The harmonic function $X[k]$ (also called Fourier series coefficients) for a periodic signal $x(t)$ with a fundamental period of $T_F = 0.1$ sec are as follows:

   $X[0] = 5$ $\quad f_F = 10\ Hz$

   $X[2] = X[-2] = ½$

   $X[4] = X[-4] = 3$

   All other $X[k] = 0$

   Write an expression for the associated time signal in terms of sinusoids.

$$x(t) = X[0]e^{j2\pi(0)f_F t} + X[2]e^{j2\pi(2)f_F t} + X[-2]e^{j2\pi(-2)f_F t} + X[4]e^{j2\pi(4)f_F t} + X[-4]e^{j2\pi(-4)f_F t}$$

$$= 5 + \underbrace{\tfrac{1}{2}e^{j2\pi 20t} + \tfrac{1}{2}e^{-j2\pi 20t}} + \underbrace{3e^{j2\pi 40t} + 3e^{-j2\pi 40t}}$$

$$\boxed{x(t) = 5 + \cos(2\pi 20t) + 6\cos(2\pi 40t)}$$

6. Find the Fourier Series Harmonic function ($X[k]$) for the function: $-3\cos(2\pi 500t)$
   $x(t) = \cos(2\pi 100t) + \sin(2\pi 150t) - 3\cos(1000\pi t)$. First, determine the fundamental period and fundamental frequency. Write your X[k] as a sum of impulses.

$$T_F = LCM\left(\frac{1}{100}, \frac{1}{150}, \frac{1}{500}\right) = \frac{LCM\left(\frac{1500}{100}, \frac{1500}{150}, \frac{1500}{500}\right)}{1500} = \frac{LCM(15, 10, 3)}{1500}$$

$$T_F = \frac{30}{1500} = \frac{1}{50} \quad \text{so } f_F = 50\ Hz$$

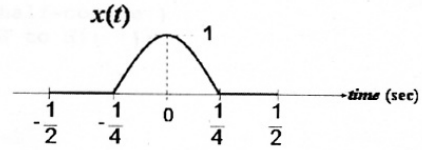
$$x(t) = \frac{1}{2}e^{j2\pi 100t} + \frac{1}{2}e^{-j2\pi 100t} + \frac{1}{2j}e^{j2\pi 150t} - \frac{1}{2j}e^{-j2\pi 150t} - \frac{3}{2}e^{j2\pi 500t} - \frac{3}{2}e^{-j2\pi 500}$$

$$x(t) = \underbrace{\tfrac{1}{2}e^{j2\pi(2)50t}}_{X[2]} + \underbrace{\tfrac{1}{2}e^{j2\pi(-2)50t}}_{X[-2]} + \underbrace{\tfrac{1}{2j}e^{j2\pi(3)50t}}_{X[3]} - \underbrace{\tfrac{1}{2j}e^{j2\pi(-3)50t}}_{X[-3]} - \underbrace{\tfrac{3}{2}e^{j2\pi(10)50t}}_{X[10]} - \underbrace{\tfrac{3}{2}e^{j2\pi(-10)50t}}_{X[-10]}$$

$$\boxed{X[k] = \tfrac{1}{2}\delta[k-2] + \tfrac{1}{2}\delta[k+2] + \tfrac{1}{2j}\delta[k-3] - \tfrac{1}{2j}\delta[k+3] - \tfrac{3}{2}\delta[k-10] - \tfrac{3}{2}\delta[k+10]}$$

0

7. For the function shown in the figure to the right,:



$\longleftarrow T_F = 1 \longrightarrow$

$f_F = 1$

a. Determine the harmonic function X[k] for this function for the time interval from t=-1/2 to t=1/2. Hint: Appendix A of the text may have a useful formula you could use:

$$\int \cos(bx)e^{ax}dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx)+b\sin(bx)\right]$$

$$X[k] = \frac{1}{T_F}\int_{t_0}^{t_0+T_F} x(t)e^{-j2\pi k f_F t}dt = \int_{-\frac{1}{4}}^{\frac{1}{4}} \cos 2\pi t\, e^{-j2\pi k t}dt$$

using Appendix A equation w/ $a = -j2\pi k$ and $b = 2\pi$

$$X[k] = \frac{1}{(-j2\pi k)^2+(2\pi)^2}\, e^{-j2\pi k t}\left[-j2\pi k\cos(2\pi t) + 2\pi\sin(2\pi t)\right]\Bigg|_{t=-\frac{1}{4}}^{\frac{1}{4}}$$

$$= \frac{1}{4\pi^2(1-k^2)}\left\{ e^{-j\pi k/2}\left[-j2\pi k \underbrace{\cos(\pi/2)}_{0} + 2\pi \underbrace{\sin(\pi/2)}_{1}\right] + \\ -e^{j\pi k/2}\left[-j2\pi k\underbrace{\cos(-\tfrac{\pi}{2})}_{0} + 2\pi\underbrace{\sin(-\pi/2)}_{-1}\right]\right\}$$

$$= \frac{1}{4\pi^2(1-k^2)}\left[e^{-j\pi k/2}(2\pi) - e^{j\pi k/2}(-2\pi)\right]$$

$$= \frac{1}{4\pi^2(1-k^2)}\,2\pi\left[e^{j\pi k/2} + e^{-j\pi k/2}\right] = \frac{\cancel{2\pi}}{\cancel{4}\pi^{\cancel{2}}(1-k^2)}\,\cancel{2}\cos\left(\frac{\pi k}{2}\right)$$

$$\boxed{X[k] = \frac{\cos\left(\frac{\pi k}{2}\right)}{\pi(1-k^2)}}$$

b. In MATLAB, create a 2x2 stem subplot of the magnitude and phase of the Harmonic function for $-10 \le k \le 10$. The phase should be in radians. Label your axes and include a grid. Turn in your code and your plot with this worksheet.

What happens when $k = \pm 1$? In this case, $X[k] = \frac{0}{0}$ = NaN (in Matlab)

- in this case use L'Hopital's Rule:

$$\left.\frac{\cos(\frac{\pi k}{2})}{\pi - \pi k^2}\right|_{k=\pm 1} = \frac{\frac{d}{dk}\cos(\pi k/2)}{\frac{d}{dk}\,\pi - \pi k^2} = \frac{-\sin(k\pi/2)\cdot(\frac{\pi}{2})}{-2\pi k}$$

This $= \frac{1}{4}$ when $k = \pm 1$

so $X[1] = X[-1] = \boxed{1/4}$

Note: Something interesting happens when you try to plot the X[1] and X[-1] term…the X[k] won't have a value for k=+1 or k=-1 unless you have a line of code that sets their value to ¼ (as the following code shows).

```matlab
t=-.5:.0001:.5;
%t=-10:.0001:10;
disp('This program will show an approximation to a half-cosine')
N=input('Enter N (will use CTFS Coefficients from -N to N): ');
%N=20;
T0 = 1;  %fundamental period
f0 = 1/T0;  %fundamental frequency
x=zeros(size(t));
index = 1;

for k=-N:N
    if (abs(k) == 1)
        FS_coeff(index) = 1/4;
      else
        FS_coeff(index) = (1/pi/(1-k^2)).*cos(k*pi/2);
    end
    x = x+FS_coeff(index) .* exp(j*2*pi*k*f0*t);
    index=index+1;
end


figure(1)
subplot(2,1,1)
N1=2*N+1;
stem(-N:N,abs(FS_coeff)),title([num2str(N1) ' CTFS Coefficients'])
xlabel('index, k (multiple of fundamental frequency)')
ylabel('magnitude'), grid on
subplot(2,1,2)
stem(-N:N,angle(FS_coeff)*180/pi)
xlabel('index, k (multiple of fundamental frequency)')
ylabel('phase(deg)'), grid on
```