EE334
Fourier and Filters Worksheet

To for most electronics, but especially for communications equipment, we often have to be able to separate different frequencies from one another. The devices that do this separation are called filters. They work in way similar to other filters that you're already familiar with.

1. Think of at least two kinds of every day filters that you might find at home. What is being separated by the filters you listed?

2. List at least two applications for electrical frequency filters.

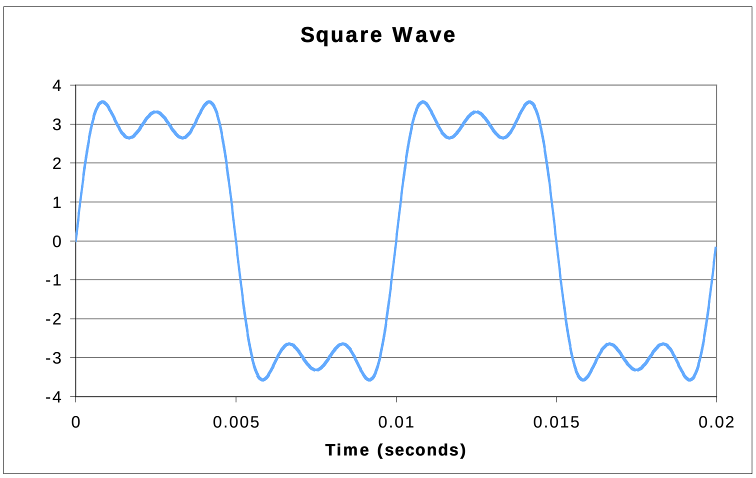

Figure 1. A square waved formed from three sine waves.

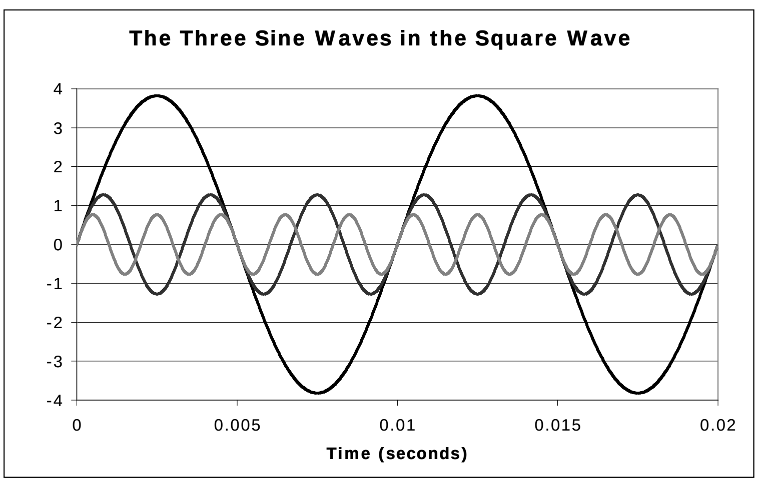

Figure 2. The three waves that when added together make the square wave in Figure 1.

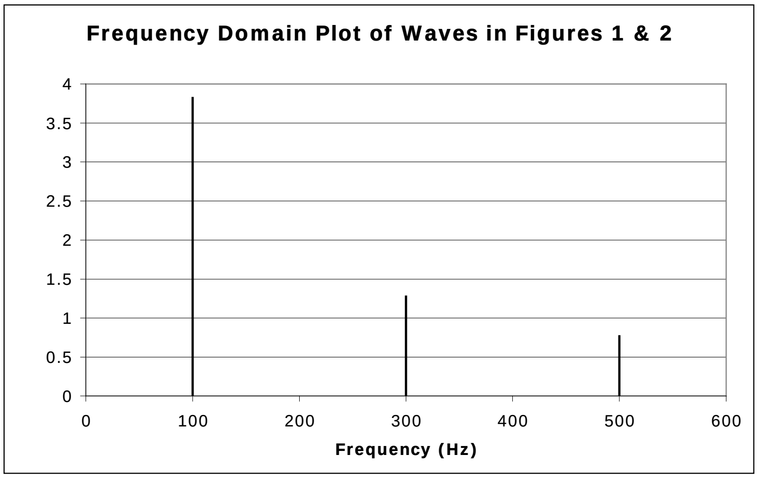

Figure 3. A frequency domain plot of the waves in Figures 1 and 2.

Figure 1 shows a square wave generated by combining three sine waves each of a different frequency and amplitude. Figure 2 shows the waves in Figure 1 before they were combined into the compound wave of Figure 1. Notice that the waves in Figures 1 and 2 are plotted in terms of volts and time. Figure 3 is a different view of the waves in Figures 1 and 2. The difference is that we're not looking at the waves in time, but rather by their frequencies. Each of the three sine waves is plotted according to its frequency and peak voltage.

FOURIER THEORY (simplified): All waveforms (everything you hear for example) can be reproduced by combining sine waves of the appropriate frequencies and amplitudes. Moreover, even waveforms that weren't created by the combination of sine waves behave as if they were. This means that complex waveforms can be broken down into their sine wave components. I.e., we can separate (filter) out certain frequencies from the mish-mash of frequencies.

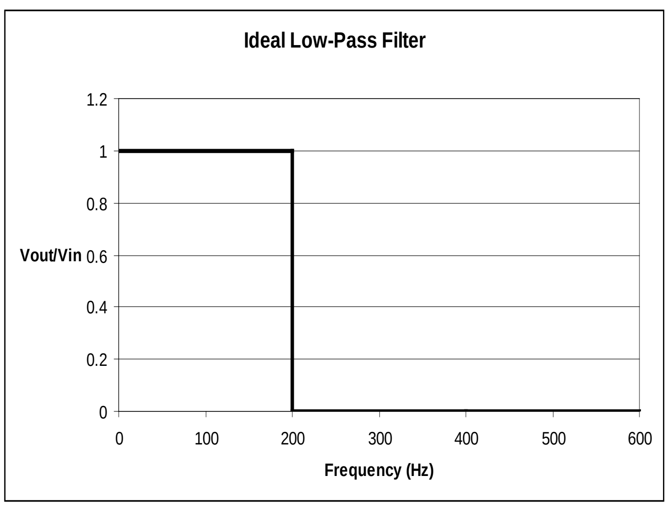

Figure 4. An ideal low-pass filter.

Figure 4 shows the frequency response curve for an ideal low-pass filter. The filter is said to be low-pass because it only lets frequencies below 200 Hz pass through it. This is reflected in

Figure 4 by the vertical axis that shows the ratio of voltage into the filter to voltage out of the filter: v_{out}/v_{in}. Frequencies higher than 200 Hz are completely blocked (attenuated) – the in/out ratio is zero past 200 Hz. The filter is ideal because the cut-off frequency f_{CO} = 200 Hz is exact. In real filters the line between passing and blocking is gradual instead of being vertical.

3. Apply the filter in Figure 4 to the compound wave of Figures 1, 2, and 3. Plot the resulting waveform in the time domain and in the frequency domain.

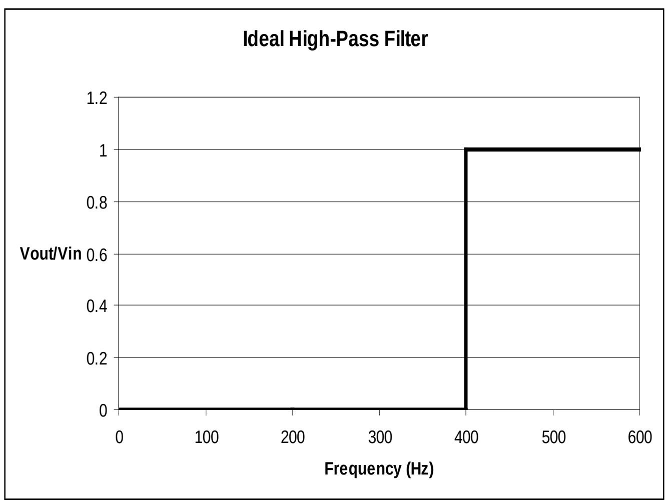

Figure 5. An ideal high-pass filter.

4. Figure 5 shows an ideal high-pass filter. Apply the high-pass filter to the compound wave of Figures 1, 2, and 3. Plot the resulting waveform in the time and frequency domains.

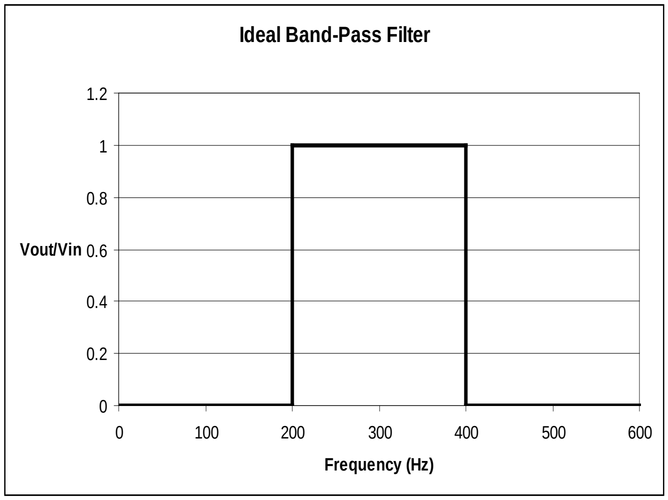

Figure 6. An ideal band-pass filter.

5. Figure 6 shows an ideal band-pass filter. The band-pass filter has two cut-off frequencies: a low cut-off f_{LOW} = 200 Hz and a high cut-off f_{HIGH} at 400 Hz. Signals with frequencies inside the band of f_{LOW} to f_{HIGH} pass through the filter ($v_{out}/v_{in} = 1$) while anything outside the band is completely filtered out ($v_{out}/v_{in} = 0$). Apply the band-pass filter to the compound wave of Figures 1, 2, and 3. Plot the resulting waveform in the time and frequency domains.

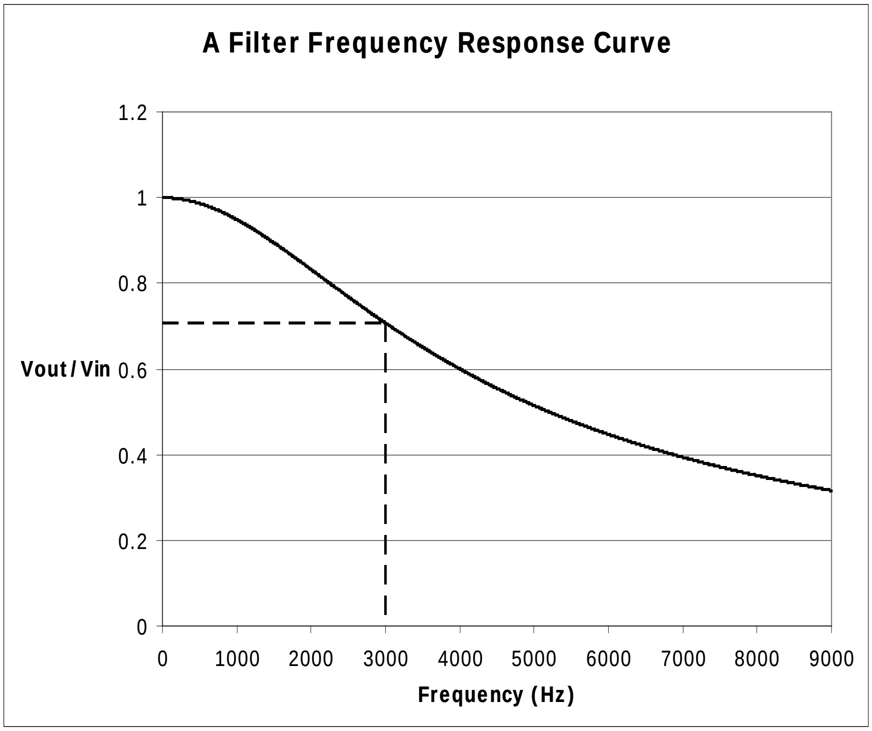

Figure 7. A frequency response curve for some filter.

6. Describe what is going on in Figure 7. Is this a very good filter? Why or why not? Use 2-3 sentences.

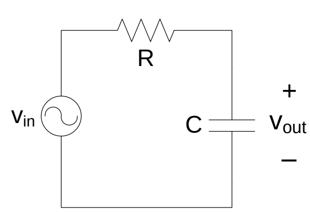

Figure 8. A filter made from a resistor and a capacitor.

The plot in Figure 7 can be made by several simple electronic circuits. The circuit in Figure 8, which consists of just a resistor and a capacitor, is one such circuit. To understand how the circuit in Figure 8 works, we need to know some facts about capacitors.

EE334
Fourier and Filters Worksheet

Capacitor Facts

- Capacitors are little batteries that are quick to charge and slower, but still quick, to discharge.
- The quickness with which a capacitor charges and discharges and how much electric charge the capacitor can store is measured in units called Faradays (Farads or F, for short).
- Like all electrical components, capacitors have resistance. The funny thing about a capacitor is that its resistance depends on the frequencies of the electric signals it experiences.
- The letter used for a capacitor is "C".
- The resistance of a capacitor is given by the following formula, where C is the capacitance of the capacitor in Farads.

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$$

- The resistance of a capacitor, Z_C, is a complex number. For now, we'll pretty much ignore the imaginary component. Your calculators make quick work of the imaginary numbers if we need to compute them.
- Recall that imaginary numbers can be represented, in parts, as phase angles. Signals passing through a capacitor have their phase angles shifted due to the capacitor's complex resistance.
- Any circuit containing just a resistor and a capacitor is, in fact, either a low-pass or high-pass filter (so-called RC filter). The voltage supply is not considered to be part of the circuit.
- The cut-off frequency for an RC filter (both low-pass and high-pass) is given by:

$$fco = \frac{1}{2\pi RC}$$

- At the cut-off frequency the following are true:

$$\frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{2}} = 0.707 \qquad \frac{P_{out}}{P_{in}} = \frac{1}{2} = 0.5$$

Let's examine the RC filter in Figure 8 and see if we can figure out how it works.

7. At what frequency is a capacitor's resistance the highest? At what frequency is a capacitor's resistance the lowest?

8. Pretend the voltage supply in Figure 8 is pumping out a signal of 0 Hz (a DC signal). Without doing any equations, how much of the voltage produced by the voltage supply will be consumed by the capacitor? How much by the resistor?

9. Now suppose the power supply is operating at ∞ Hz. How much of the power supply's voltage, v_{in}, will be consumed by the capacitor (how much of v_{in} makes it to v_{out})? How much of v_{in} is consumed by the resistor?

10. If we hooked up a voltmeter (a device that measures voltage) to v_{out}, would the voltmeter register a lot of voltage when the power supply is at 0 Hz or at ∞ Hz?

11. Since we can measure __________ frequency voltage changes at v_{out}, this particular circuit configuration is passing __________ frequency signals to the voltmeter (or any other voltage sensitive device we might hook up at v_{out}) and is said to be a __________-pass filter.

12. Express v_{out} (write an equation) as a function of v_{in}, R, and Z_C. Hint: Think of the circuit in Figure 8 as being a voltage divider and treat the capacitor as a resistor.

13. Modify the previous equation to be of the form v_{out}/v_{in}. This is called the <u>transfer function</u> or H(f) for short.

$$H(f) = \frac{v_{out}}{v_{in}}(f) =$$

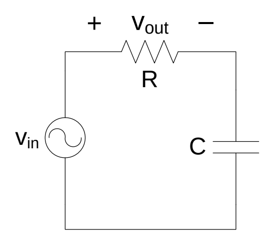

Figure 9. Another RC filter.

14. Look at Figure 9. It's the same as Figure 8, but v_{out} has been moved to be across the resistor. Derive the transfer function for the circuit in Figure 9.

15. Look at the transfer function you just derived. What is the ratio of v_{out} to v_{in} when the power supply's frequency is 0 Hz (f = 0 Hz)? What is the ratio when $f = \infty$?

16. Hmmmm. So this is the same circuit except that the location that we are considering the output, v_{out}, has changed. Yet the transfer function and behavior of the circuit is different. In fact, from the viewpoint of the new v_{out} this is now a _____________-pass filter!

17. Draw a frequency response curve (in the fashion of Figure 7) for the circuit in Figure 9. A sketch is OK.

Figure 10. Generalized frequency response curve for the circuit in Fig. 9.

18. The facts section gives an equation for the cut-off frequency for RC filters. Substitute $f = f_{co}$ into the transfer functions you derived for parts #13 and #14. Any surprises?

19. Look at Figure 7. Notice anything about the dashed lines on the diagram and the transfer function when f = f_{co}? Add similar dashed lines to your curve in Fig 10.

20. f_{co} is also called the "half-power frequency" because when a filter is driven at f_{co} (when f = f_{co}) then the power at the output of the circuit is ½ of the power at the input (P_{out} = ½ P_{in}). Derive P_{out}. Here are a few facts to get you started.
    a. We already saw that when f = f_{co} that $v_{out} = \frac{v_{in}}{\sqrt{2}}$
    b. $P_{out} = \frac{v_{out}^2}{R}$
    c. $P_{in} = \frac{v_{out}^2}{R}$

Inductor Facts

- Like a capacitor, an inductor is a device that stores energy and has a resistance that changes with frequency. One difference is that while a capacitor stores electrical charge an inductor stores energy as an electric field.
- The letter used for an inductor is “L”.
- On circuit diagrams inductors are drawn as springs or coils of wire. This is because some inductors are built from wire windings.

- The storage capacity of an inductor is measured in Henries (H).
- The resistance of an inductor is given by the following formula where L is the inductance of the inductor in Henries:

$$Z_L = j\omega L + R_L = j2\pi fL + R_L$$

- R_L in the formula above is usually ignored in the kinds of problems we’ll work ($R_L \approx 0\ \Omega$).
- Since inductors are frequency sensitive, we can use them in filters.
- Any circuit containing just a resistor and an inductor is, in fact, either a low-pass or high-pass filter (so-called RL filter). The voltage supply is not considered to be part of the circuit.
- The cut-off frequency for an RL filter is given by:

$$f_{co} = \frac{R}{2\pi L}$$

- At the cut-off frequency the following are true (same as for an RC filter):

$$\frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{2}} = 0.707 \qquad \frac{P_{out}}{P_{in}} = \frac{1}{2} = 0.5$$

21. At what frequency is an inductor’s resistance the lowest? When is it the highest? How do these answers compare to a capacitor’s resistance extremes?

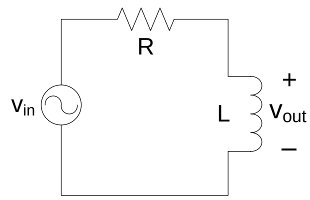

Figure 11. An RL filter.

22. Derive the transfer function, H(f), for the circuit in Figure 11.

23. Draw a frequency response graph for the circuit of Figure 11. Use the transfer function from #22 to help you. Make sure you label axes and indicate f_{co}. A rough sketch is fine.

24. Is this a low-pass or high-pass filter? Why?

25. Based on what you saw with RC filters, what kind of filter do you think would result if we moved v_{out} from the inductor to the resistor? Why do you think this? No need to do math.

26. Redraw the circuit of Figure 11 below as Figure 12 but move v_{out} to the resitor.

Figure 12. Another RL filter.

27. Derive H(f) for the circuit in you just drew as Figure 12.

28. Draw a frequency response curve for the transfer function. Label axes. A sketch is fine.

29. Substitute $f = f_{co} = \frac{R}{2\pi L}$ into the transfer functions you derived in #22 and #27. What is significant about the results? How do these results compare to the transfer functions for the RC filters when $f = f_{co}$?

30. You are presented with an RC filter and told that R = 2.5k Ω and that C = 4.7n F. What is the cut-off frequency for this filter? TRICK: Is this filter high-pass or low-pass?

31. You are told to create a low-pass filter using an inductor and a resistor. The resistor has a value of 5.6k Ω. The cut-off frequency needs to be 500k Hz. What size of inductor should you use? What size inductor would you use to create a high-pass filter using the same sized resistor?

Notice that the transfers function we derived are complex – they both real and imaginary components. On your calculator, you can plug in the transfer function and see the result in polar coordinates (a magnitude and a phase angle) or as a complex number (real part and the imaginary part). The polar version of the complex number is most useful in this class. In the case of the transfer function (this one and others we'll see later), the magnitude of the polar number is the fraction of voltage passed through the filter. Interestingly, the phase angle tells us how far forward or backwards in time the waveform shifts. The formulas below are a shortcut to computing the magnitude of the transfer functions for low-pass and high-pass filters. The low-pass formula was used to generate the plot in Figure 7.

Low-pass filter transfer function magnitude: $\|H(f)\| = \left\|\frac{v_{in}}{v_{out}}\right\| = \frac{1}{\sqrt{1+\left(\frac{f}{f_{co}}\right)^2}}$

High-pass filter transfer function magnitude: $\|H(f)\| = \left\|\frac{v_{in}}{v_{out}}\right\| = \frac{\frac{f}{f_{co}}}{\sqrt{1+\left(\frac{f}{f_{co}}\right)^2}}$

32. Complete the table below concerning the resistances of capacitors and inductors at frequency extremes.

Circuit subjected to ⇒	Extreme low freq (0 Hz)	Extreme high freq (∞ Hz)
Capacitor	Z_C =	Z_C =
Inductor	Z_L =	Z_L =

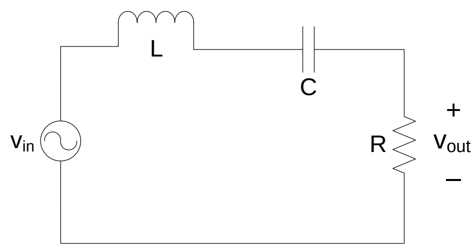

Figure 13. A new type of filter.

33. Examine the circuit in Figure 13. It's a kind of filter. Make and educated guess about what frequencies produced by the power supply at v_{in} will make it to v_{out} without being attenuated too much. **Hint:** Look at the low-pass and high-pass RC and RL filters where v_{out} was measured across the resistor. Think of superimposing those filters on top of each other to produce the one in Figure 13.

34. Derive the transfer function for the circuit in Figure 13.

35. Recall that the transfer function tells us how much of v_{in} makes it (passes through) to v_{out}. Check the guess made in #33 by plugging in frequencies of 0 Hz and ∞ Hz into the transfer function from #34.

36. Let's further confirm the guess from #33 by examining the transfer function a little bit closer. We know that the transfer function has a range of 0 to 1 (none of v_{in} makes it to v_{out} up to all of v_{in} makes it to v_{out}, respectively). We're interested in finding the frequency that enables all of v_{out} to transfer to v_{in}. This happens when the resistor is the only resistance in the circuit due to the inductor and capacitor resistances cancelling each other out. That is, when $Z_L - Z_C = 0\Omega$ then the transfer function simplifies to $H(f) = \frac{R}{R} = 1$. Teach each other.

37. Using the fact that $Z_L + Z_C = 0\Omega$ maximizes the transfer function, solve for the frequency that produces this maximum value. **Hint:** Start with $Z_L + Z_C = 0\ \Omega$ and solve for frequency.

38. Sketch the frequency response curve for the circuit in Figure 13. Make it big and leave some room for adding additional labels and points of interest.

39. What kind of filter is this? Hint: You saw an ideal version earlier in this packet.

40. The frequency where the ratio v_{out}/v_{in} is highest is called the center frequency, f_c. Label f_c on the graph in #38.

41. There are two cut-off frequencies for this filter. One is on the low frequency side of fc and is usually called f_{LOW}. The other is on the high side and is called f_{HIGH}. Label f_{LOW} and f_{HIGH} on the graph in #38. Make sure you indicate the value of v_{out}/v_{in} at the cut-off frequencies.

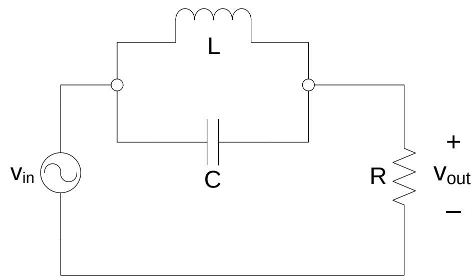

Figure 14. Another mysterious filter.

42. Let's test your skills. Quickly figure out the behavior of the filter shown in Figure 14. You may redraw the circuit, sketch frequency response curves, or derive and solve equations. We'll assigned teams to work on each method and then compare answers later.

43. How does this filter compare to other filters you've seen so far? Especially band-pass?

Band-Pass and Notch-Pass Filter Facts

- The frequency which v_{out}/v_{in} is maximized (minimized for notch-pass) is the center frequency, f_c, and is given by the formula:

$$f_c = \frac{1}{2\pi\sqrt{LC}}$$

- There are two cut-off frequencies, one on either side of fc, called f_{LOW} and f_{HIGH}.
- The <u>bandwidth</u> (BW) of a bandpass filter is the span of frequencies between f_{LOW} and f_{HIGH}.

$$BW = f_{HIGH} - f_{LOW}$$

- The ratio of fc to BW gives you an idea of how narrow or wide the filter is (the concept of narrow and wide depends on the center frequency) and has a special name called the Q-value or just Q. Important formulas involving Q are:

$$Q = \frac{f_c}{BW} = \frac{2\pi f_c L}{R} = \frac{1}{2\pi f_c CR}$$

- When Q is high (Q > 10), as it is for most useful band-pass filters, then fc is roughly centered between f_{LOW} and f_{HIGH}:

$$f_c = \frac{f_{LOW} + f_{HIGH}}{2}$$

- At the cut-off frequencies the following are true (same as for an RC or RL filter):

$$\frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{2}} = 0.707 \qquad \frac{P_{out}}{P_{in}} = \frac{1}{2} = 0.5$$

If time permits, do some homework problems concerning filters either in teams or on your own.