EE362
Babbage's Difference Engine
Example to Compute Squares of Integers

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1. Show how to compute $y = x^2$ using the approach Babbage used in programming his difference engine.

   SOLUTION: For every positive integer i we have

$$y_j = x_j^2. \tag{1}$$

   Therefore

$$\begin{aligned} y_{j+1} &= (x_j + 1)^2 \\ y_{j+1} &= x_j^2 + 2x_j^2 + 1, \end{aligned} \tag{2}$$

   and

$$\begin{aligned} y_{j+2} &= (x_j + 2)^2 \\ y_{j+2} &= x_j^2 + 4x_j^2 + 4. \end{aligned} \tag{3}$$

   We can compute the first differences as

$$\begin{aligned} \Delta^1 y_j &= y_{j+1} - y_j \qquad (4) \\ &= x_j^2 + 2x_j + 1 \\ &\quad - x_j^2 \\ \Delta^1 y_j &= 2x_j + 1, \qquad (5) \end{aligned}$$

   and

$$\begin{aligned} \Delta^1 y_{j+1} &= y_{j+2} - y_{j+1} \qquad (6) \\ &= x_j^2 + 4x_j^2 + 4 \\ &\quad - x_j^2 - 2x_j^2 - 1 \\ \Delta^1 y_j &= 2x_j + 3. \qquad (7) \end{aligned}$$

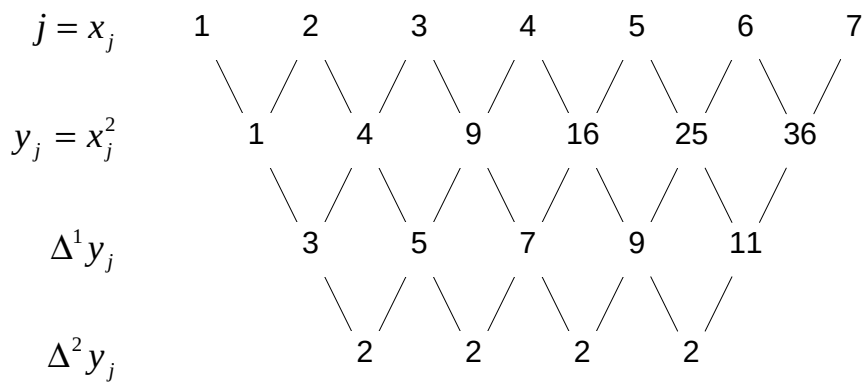

Figure 1: Computation of $y_j = x_j^2$ in Babbage's Difference Engine

We can also compute the second difference as

$$\begin{aligned} \Delta^2 y_j &= \Delta^1 y_{j+1} - \Delta^1 y_j && (8) \\ &= 2x_j + 3 \\ &\quad - 2x_j - 1 \\ \Delta^2 y_j &= 2. && (9) \end{aligned}$$

There is no need to compute higher-order differences: they all will be zero.

Using equation (8) we can write

$$\Delta^1 y_{j+1} = \Delta^2 y_j + \Delta^1 y_j. \tag{10}$$

$$\tag{11}$$

Using equation (4) we can write

$$y_{j+1} = \Delta^1 y_j + y_j. \tag{12}$$

For $j = 1$, we have $x_j = x_1 = 1$ and $y_j = y_1 = x_1^2 = 1$.

Using equation (5), $\Delta^1 y_1 = 2x_j + 1 = 3$.

Using equation (9), $\Delta^2 y_1 = 2$.

These initial values appear in the lefthand side of Figure 1. All other values are calculated using equations (12), and (10).

Many functions can be approximated by polynomials. This method covers all such functions.