Two's Complement Multiplication

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Consider the problem of multiplying two two's-complement numbers together. An n-bit two's complement number x can be expressed

$$x = x_{n-1}x_{n-2}\ldots x_2x_1x_0. \tag{1}$$

Its negative can be computed by

$$-x = \left(\overline{x}_{n-1}\overline{x}_{n-2}\ldots\overline{x}_2\overline{x}_1\overline{x}_0 + \overbrace{00\ldots001}^{n \text{ bits}}\right) \mod 2^n \tag{2}$$

$$= \left(\overbrace{11\ldots111}^{n \text{ ones}} - x_{n-1}x_{n-2}\ldots x_2x_1x_0 + \overbrace{00\ldots001}^{n \text{ bits}}\right) \mod 2^n. \tag{3}$$

There are two cases when $x \neq 0$, one for the case where $x > 0$ and the other for the case where $x < 0$.
Case i: $x > 0$

$$-x = \left(\overbrace{11\ldots111}^{n \text{ ones}} - 0x_{n-2}\ldots x_2x_1x_0 + \overbrace{00\ldots001}^{n \text{ bits}}\right) \mod 2^n \tag{4}$$

$$= (2^n - 0x_{n-2}\ldots x_2x_1x_0) \mod 2^n \tag{5}$$

$$= -0x_{n-2}\ldots x_2x_1x_0. \tag{6}$$

Case ii $x < 0$

$$
\begin{aligned}
-x &= \left(\overbrace{11\ldots111}^{n \text{ ones}} - 1x_{n-2}\ldots x_2x_1x_0 + \overbrace{00\ldots001}^{n \text{ bits}}\right) \mod 2^n && (7)\\
&= \left(\overbrace{01\ldots111}^{n \text{ bits}} - 0x_{n-2}\ldots x_2x_1x_0 + \overbrace{00\ldots001}^{n \text{ bits}}\right) \mod 2^n && (8)\\
&= \left(\overbrace{10\ldots000}^{n \text{ bits}} - 0x_{n-2}\ldots x_2x_1x_0\right) \mod 2^n && (9)\\
&= \left(2^{n-1} - 0x_{n-2}\ldots x_2x_1x_0\right) \mod 2^n && (10)\\
&= 2^{n-1} - 0x_{n-2}\ldots x_2x_1x_0. && (11)
\end{aligned}
$$

Combining (6) and (11), we have

$$
-x = 2^{n-1}x_{n-1} - \sum_{i=0}^{n-2} x_i 2^i \tag{12}
$$

Taking the negative of this shows that

$$
x = -2^{n-1}x_{n-1} + \sum_{i=0}^{n-2} x_i 2^i. \tag{13}
$$

If x is the multiplier and y is the multiplicand. Each bit x_i of x is multiplied by y and by its own weighting factor 2^i or, in the case of bit x_{n-1}, by -2^{n-1}. In other words, if the multiplier x is negative, the $(n-1)$-th bit's product is *subtracted* from the overall product. This is the basis of the Robertson multiplication algorithm.