C3M14

Notes on Line Integrals

I. Some Basic Parameterizations.

The successful completion of problems involving line integrals often depends on one's ability to set up the vector functions, or spacecurves, to parameterize the paths involved. There are several basic types that we will need:

A. **Straight line joining two given points**. Suppose we wish to traverse the line segment from $P(2,-1,3)$ to $Q(-3,5,-1)$. In generic form we have

$$\vec{\alpha}(t) = (1-t)P + tQ \qquad 0 \le t \le 1$$

and specifically, for $0 \le t \le 1$

$$\begin{aligned}\vec{\alpha}(t) &= (1-t)\langle 2,-1,3\rangle + t\langle -3,5,-1\rangle \\ &= \langle 2-2t,\ -1+t,\ 3-3t\rangle + \langle -3t,\ 5t,\ -t\rangle \\ &= \langle 2-5t,\ -1+6t,\ 3-4t\rangle\end{aligned}$$

Note that $\vec{\alpha}(0) = P$ and $\vec{\alpha}(1) = Q$.

B. **Circle of radius a at origin traversed counterclockwise**. Use your polar coordinate experience and define

$$\vec{\alpha}(t) = \langle a\cos t,\ a\sin t\rangle \qquad 0 \le t \le 2\pi$$

And we may translate this circle to a new center (h,k) by adding $\langle h,k\rangle$.

$$\vec{\alpha}(t) = \langle h + a\cos t,\ k + a\sin t\rangle \qquad 0 \le t \le 2\pi.$$

C. **Ellipse traversed counterclockwise**. If the equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then use

$$\vec{\alpha}(t) = \langle a\cos t,\ b\sin t\rangle \qquad 0 \le t \le 2\pi.$$

D. **Elbow path**. These are handled in two parts. One begins each leg by identifying which variable does not change. For example, join $(2,1) \to (2,5) \to (-4,5)$. The first leg has $x = 2$ and one inserts that fact first. Then the other coordinate is handled naturally.

$$\vec{\alpha}_1(t) = \langle 2,\quad \rangle = \langle 2, t\rangle \qquad 1 \le t \le 5$$

The second leg is given by $\vec{\alpha}_2(t) = \langle\quad, 5\rangle = \langle 2-6t, 5\rangle \qquad 0 \le t \le 1$.

II. Some remarks concerning line integrals.

To begin our discussion, let's assume that a path C has been parametrized by a continuous function $\vec{\alpha} : [a,b] \to \mathbb{R}^2$, i.e. $\vec{\alpha}(t) = \langle x(t), y(t)\rangle$ on $[a,b]$. Also assume $\vec{F} = \langle M, N\rangle$ is a vector-valued function defined on a region that contains C. So $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$. If we let $\vec{r} = \langle x,\ y,\rangle$ then $d\vec{r} = \langle dx,\ dy\rangle$. This is just another way of writing $\vec{\alpha}'(t) = \langle x'(t), y'(t)\rangle$.

The line integral of $\vec{F}$ over the path C is defined by

$$\int_C M\,dx + N\,dy = \int_C \langle M,N\rangle \cdot \langle dx, dy\rangle = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{\alpha}(t))\cdot\vec{\alpha}'(t)\,dt$$

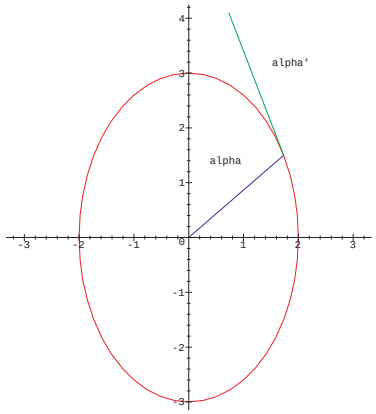
If $\vec{T}(t) = \vec{\alpha}'(t)/\|\vec{\alpha}'(t)\|$ and $\frac{ds}{dt} = \|\vec{\alpha}'(t)\|$ so that $ds = \|\vec{\alpha}'(t)\|\,dt$, then

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{\alpha}(t))\cdot\frac{\vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\|}\,\|\vec{\alpha}'(t)\|\,dt = \int_a^b \vec{F}(\vec{\alpha}(t))\cdot\vec{\alpha}'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

We mention this just to remind you that there are several notations that refer to the same mathematical entity. But the form we will use most often to evaluate line integrals is

$$\int_a^b \vec{F}(\vec{\alpha}(t))\cdot\vec{\alpha}'(t)\,dt$$

Example 1 Suppose $\vec{F}(x,y) = \left\langle \dfrac{-y}{\sqrt{9x^2+4y^2}}, \dfrac{x}{\sqrt{9x^2+4y^2}} \right\rangle$ and C is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ traversed counterclockwise. Evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.



On the interval $0 \le t \le 2\pi$ we have $\vec{\alpha}(t) = \langle 2\cos(t),\, 3\sin(t)\rangle$ and $\vec{\alpha}'(t) = \langle -2\sin(t),\, 3\cos(t)\rangle$. On C $9x^2 + 4y^2 = 9 \cdot 4\cos^2(t) + 4 \cdot 9\sin^2(t) = 36$, so

$$\vec{F}\big(\vec{\alpha}(t)\big) = \left\langle \frac{-3\sin(t)}{\sqrt{36}}, \frac{2\cos(t)}{\sqrt{36}} \right\rangle = \left\langle \frac{-3\sin(t)}{6}, \frac{2\cos(t)}{6} \right\rangle$$

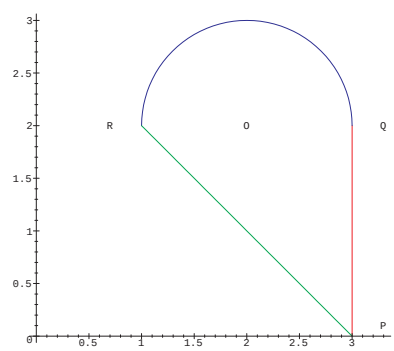
The next step is to form

$$\begin{aligned} \vec{F}\left(\vec{\alpha}(t)\right) \cdot \vec{\alpha}'(t) &= \left\langle \frac{-\sin(t)}{2}, \frac{\cos(t)}{3} \right\rangle \cdot \langle -2\sin(t),\, 3\cos(t)\rangle \\ &= \sin^2(t) + \cos^2(t) = 1 \end{aligned}$$

Putting this together we get

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1\, dt = 2\pi$$

Example 2 (Maple) Suppose $\vec{F}(x,y) = \langle -y, x\rangle$ and C is the closed path consisting of the vertical line from $P(3,0)$ to $Q(3,2)$, followed by the semicircular arc centered at $O(2,2)$ that ends at $R(1,2)$ traversed counterclockwise, followed by the straight line from R to P.



Because the path C has three distinct parts, we have three line integrals whose values are added together to find the solution.

Path C_1: Line segment from $P(3,0)$ to $Q(3,2)$. Use $\vec{\alpha}_1(t) = \langle 3, 2t\rangle$ for $0 \le t \le 1$.

```
> restart:    with(linalg):    with(student):
> F:=(x,y)->[-y,x];
```

$$F := (x,y) \to [-y, x]$$

```
> alpha1:=[3,2*t];
```

$$\alpha 1 := [3, 2t]$$

```
> Falpha1:=F(op(alpha1));
```

$$Falpha1 := [-2t, 3]$$

```
> alpha1prime:=map(diff,alpha1,t);
```

$$alpha1prime := [0, 2]$$

```
> grand1:=innerprod(Falpha1,alpha1prime);
```

$$grand1 := 6$$

```
> Lint1:=Int(grand1,t=0..1);
```

$$Lint1 := \int_0^1 6\, dt$$

```
> L1:=value(Lint1);
```

$$L1 := 6$$

Path C_2: Semicircular arc, radius 1, center at $O(2,2)$. Use $\vec{\alpha}_2(t) = \langle \cos(t)+2, \sin(t)+2\rangle$ for $0 \le t \le \pi$.

```
> alpha2:=[cos(t)+2,sin(t)+2];
```

$$\alpha 2 := [\cos(t)+2, \sin(t)+2]$$

```
> alpha2prime:=map(diff,alpha2,t);
```

$$alpha2prime := [-\sin(t), \cos(t)]$$

```
> Falpha2:=F(op(alpha2));
```

$$Falpha2 := [-\sin(t)-2, \cos(t)+2]$$

```
> grand2:=innerprod(Falpha2,alpha2prime);
```

$$grand2 := \sin(t)^2 + 2\sin(t) + \cos(t)^2 + 2\cos(t)$$

```
> grand2:=simplify(grand2,symbolic);
```

$$grand2 := 2\sin(t) + 2\cos(t) + 1$$

```
> Lint2:=Int(grand2,t=0..Pi);
```

$$Lint2 := \int_0^{\pi} 2\sin(t) + 2\cos(t) + 1\, dt$$

```
> L2:=value(Lint2);
```

$$L2 := 4 + \pi$$

Path C_3: Line segment from $R(1,2)$ to $P(3,0)$. Use $\vec{\alpha}_3(t) = \langle t, 3-t\rangle$ for $1 \le t \le 3$.

```
> alpha3:=[t,3-t];
```

$$\alpha 3 := [t, 3-t]$$

```
> alpha3prime:=map(diff,alpha3,t);
```

$$alpha3prime := [1, -1]$$

```
> Falpha3:=F(op(alpha3));
```

$$Falpha3 := [-3+t, t]$$

```
> grand3:=innerprod(Falpha3,alpha3prime);
```

$$grand3 := -3$$

```
> Lint3:=Int(grand3,t=1..3);
```

$$Lint3 := \int_1^3 -3\, dt$$

```
> L3:=value(Lint3);
```

$$L3 := -6$$

Combining our answers we get

```
> Lineint:=L1+L2+L3;
```

$$Lineint := 4 + \pi$$

Because C was a closed path, we will revisit Example 2 when we discuss Green's Theorem.

III. Line integrals involving conservative vector fields.

When this becomes applicable we will have discussed conservative vector fields, path independence, and the curl of a vector field quite thoroughly. In any case, it may help to read C3M13 again. In the theorem stated there it is useful to note that part (a) is based on integrating along an elbow path $(x_0, y_0) \to (x, y_0) \to (x, y)$. Part (b) uses $(x_0, y_0, z_0) \to (x, y_0, z_0) \to (x, y, z_0) \to (x, y, z)$.

Example 2 of C3M13, revisited Evaluate $\int_C \vec{F} \cdot d\vec{r}$ if

$$\vec{F}(x,y) = \left\langle 3x^2\cos(2y) + \frac{y}{x},\ -2x^3\sin(2y) + \ln x \right\rangle$$

and C is the polygonal path $(1, \pi/4)$ to $(4, \pi/6)$ to $(3, \pi/3)$.

We note that $P = (1, \pi/4)$ and $Q = (3, \pi/3)$ and recall that $g(x, y) = x^3 \cos(2y) + y \ln x$ is a potential function for $\vec{F}$.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= g(Q) - g(P) = g(3, \pi/3) - g(1, \pi/4) \\ &= \Big(27 \cos\big(\frac{2\pi}{3}\big) + \frac{\pi}{3} \ln 3\Big) - \Big(\cos\big(\frac{\pi}{2}\big) + \frac{\pi}{4} \ln 1\Big) \\ &= \frac{-27}{2} + \frac{\pi}{3} \ln 3\end{aligned}$$

The theorem that follows sums up conservative vector fields and line integrals quite nicely.

THEOREM: Suppose $\vec{F}$ is a vector field with all relevant components and their partial derivatives continuous on an open simply connected and connected region D. Then the following are equivalent (TFAE):

(A) $\vec{F}$ is conservative.

(B) $\vec{F}$ is path independent.

(C) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C.

(D) The curl of $\vec{F}$, $\nabla \times \vec{F} = \vec{0}$ at all points of D.

Further, if g is a potential for $\vec{F}$ on D and C is a path with initial point P and terminal point Q, then

$$\int_C \vec{F} \cdot d\vec{r} = g(Q) - g(P).$$

EXERCISES

I. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ over the given path.

1. $\vec{F}(x, y) = \langle 3y/x,\, 5x \rangle$ and C is parametrized by $\vec{\alpha}(t) = \langle t^2,\, t^3 \rangle$ for $1 \le t \le 2$.

2. C is the straight line segment from $P(-1, 3, 1)$ to $Q(1, 2, 3)$, find

$$\int_C y\, dx - z\, dy + 2y\, dz$$

3. C is the path that traverses the circle $x^2 + y^2 = 1$ from $P(0, -1)$ to $Q(-1, 0)$ in the counterclockwise direction. Evaluate $\int_C x^2\, dx + y\, dy$.

4. C is the straight line path from $P(2, -1, 3)$ to $Q(-1, 1, 1)$. $\vec{F}(x, y, z) = \langle x - y - z,\, 2x + y - 2z,\, x + y \rangle$.

II. Determine whether or not the given integral is path independent.

$$\int_P^Q (2y^3)\, dx + (6xy^2 + z^2)\, dy + (2yz)\, dz$$

III. For the functions given:

(A) Prove that $\vec{F}$ is path independent

(B) Find a potential function for $\vec{F}$

(C) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the path C

1. $\vec{F}(x, y) = \langle 4x^3 \ln y + y,\, \frac{x^4}{y} + x \rangle$ with C the polygonal path $(1, 1)$ to $(5, 2)$ to $(3, e)$.

2. $\vec{F}(x, y) = \langle 6x^2 \ln y,\, \frac{2x^3}{y} + 1 \rangle$, C the polygonal path $(0, 1)$ to $(1, 2)$ to $(4, 3)$

3. $\vec{F}(x, y) = \big\langle 4\cos(2x)\ln(y) + 6xy^3 + 1,\, \frac{2\sin(2x)}{y} + 9x^2y^2 \big\rangle$ with C the line segment from $(\pi/6, 1)$ to $(\pi/4, 2e)$.

4. $\vec{F}(x, y, z) = \langle 2xy + \cos(\pi z),\, x^2 + z,\, y - \pi x \sin(\pi z) \rangle$ with C the line segment from $P(1, 1, 1)$ to $Q(2, 2, 2)$.