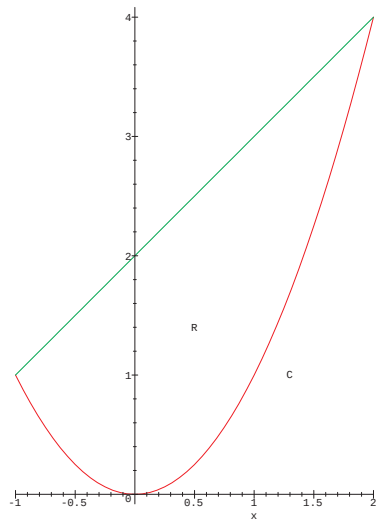
C3M15

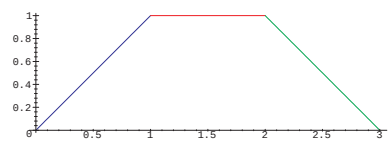
Notes on Line Integrals - Green's Theorem

GREEN'S THEOREM: Let R be a simple region in the xy-plane with a piecewise smooth boundary C that is oriented counterclockwise. Let $\vec{F}$ be a vector field with all relevant components and their partial derivatives continuous on an open region containing R. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\,dA$$



Example 1 Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y) = \langle 12x^2 \sin y + 3xy^2,\ 4x^3 \cos y + 6x^2 y\rangle$ around the polygonal path $(0,0) \to (3,0) \to (2,1) \to (1,1) \to (0,0)$.



We form the integrand for the double integral in Green's Theorem.

$$\frac{\partial N}{\partial x} = 12x^2 \cos y + 12xy \quad \text{and} \quad \frac{\partial M}{\partial y} = 12x^2 \cos y + 6xy \quad \Longrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy$$

It follows from Green's Theorem that

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\,dA \\
&= \int_0^1 \int_y^{3-y} 6xy\,dx\,dy \\
&= \int_0^1 3x^2 y\Big|_y^{3-y}\,dy = \int_0^1 3(3-y)^2 y - 3y^3\,dy \\
&= \int_0^1 27y - 18y^2\,dy = \frac{27}{2}y^2 - 6y^3\Big|_0^1 \\
&= \frac{15}{2}
\end{aligned}$$

Example 2 (Example 2 of C3M14 revisited) The vector-valued function was $\vec{F}(x,y) = \langle -y, x\rangle$. Please refer back for a diagram of the region. With $M(x,y) = -y$ and $N(x,y) = x$,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$$

So we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 2\, dA = (2)\text{area}(R) = (2)\left(\frac{\pi}{2} + 2\right) = \pi + 4$$

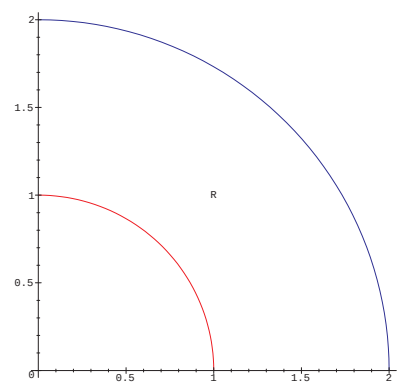
The computation of the area was easy - one half of a disk of radius 1 and one half of a square of side 2.

Example 3 The region R is that portion of the first quadrant between the circles of radius 1 and 2 centered at the origin. The vector-valued function is

$$\vec{F}(x, y) = \langle 4 + e^{\sqrt{x}}, \sin(y) + 3x^2 \rangle$$

and the objective is to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ if C is the boundary of R traversed in a counterclockwise manner. It is not easy to evaluate the line integral around this path, so our approach is to use Green's theorem. We also observe that polar coordinates provides the simplest double integral. First, we will display the region.

```
> with(student):    with(plots):
> Q1:=polarplot([1,t,t=0..Pi/2]):    Q2:=polarplot([2,t,t=0..Pi/2]):
> T:=textplot([1,1,`R`]))
> display(Q1,Q2,T);
```




```
> F:=(x,y)->[4+exp(sqrt(x)),sin(y)+3*x^2];
```

$$F := (x, y) \to [4 + e^{\sqrt{x}}, \sin(y) + 3x^2]$$

```
> Nx:=diff(F(x,y)[2],x);
```

$$Nx := 6x$$

```
> My:=diff(F(x,y)[1],y);
```

$$My := 0$$

```
> grand:=Nx-My;
```

$$grand := 6x$$

```
> grand:=subs(x=r*cos(t),grand);
```

$$grand := 6\, r\, \cos(t)$$

```
> Ans:=Doubleint(grand*r,r=1..2,t=0..Pi/2);
```

$$Ans := \int_0^{\pi/2} \int_1^2 6r^2 \cos(t)\, dr\, dt$$

```
> Greenans:=value(Ans);
```

$$Greenans := 14$$

Example 4 If $\vec{F}(x, y) = \langle -y/2, x/2 \rangle$, then $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = \dfrac{1}{2} - \left(-\dfrac{1}{2}\right) = 1$ and

$$\iint_R 1\, dA = \text{area}(R)$$

gives us a way of using line integrals to determine the area of a region. If the objective is to find the area of a region whose boundary is reasonable to parameterize, then **choose** the vector-valued function $\vec{F}(x, y) = \langle -y/2, x/2 \rangle$ and apply Green's Theorem by evaluating the resulting line integral.

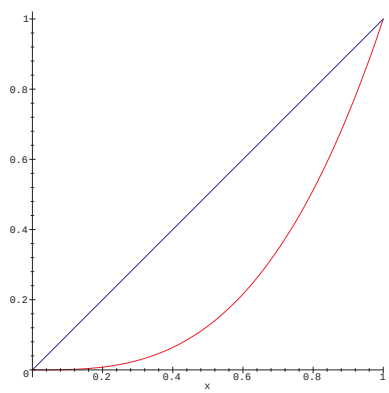
Suppose we find the area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. We parametrize C in the counterclockwise direction by $\vec{\alpha}(t)=\langle a\cos t, b\sin t\rangle$ for $0\le t\le 2\pi$. With $\vec{F}(x,y)=\langle -y/2, x/2\rangle$ we have

$$\begin{aligned}
\vec{F}\big(\vec{\alpha}(t)\big) &= \langle -\frac{b}{2}\sin t,\ \frac{a}{2}\cos t\rangle \\
\vec{\alpha}'(t) &= \langle -a\sin t,\ b\cos t\rangle \\
\vec{F}\big(\vec{\alpha}(t)\big)\cdot\vec{\alpha}'(t) &= \frac{ab}{2}\sin^2 t+\frac{ab}{2}\cos^2 t=\frac{ab}{2} \\
\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi}\frac{ab}{2}\,dt \\
&= \pi ab
\end{aligned}$$

Thus, the area of the ellipse is πab. What happens if $a=b$?

Example 5 Verify Green's Theorem for $\vec{F}(x,y)=\langle y^3, x^3+3xy^2\rangle$ and the region R which lies between $y=x$ and $y=x^3$.

```
> with(student):    with(plots):    with(linalg):
> plot([x^3,x]x=0..1);
```




```
> M:=(x,y)->y^3;  N:=(x,y)->x^3+3*x*y^2;
```

$$M := (x,y)\to y^3$$
$$N := (x,y)\to x^3+3xy^2$$

```
> F:=(x,y)->[M(x,y),N(x,y)];
```

$$F := (x,y)\to [M(x,y), N(x,y)]$$

Let's begin by evaluating the line integral along the cubic path.

```
> alpha:=[t,t^3];
```

$$\alpha := [t, t^3]$$

```
> Falpha:=F(op(alpha));
```

$$Falpha := [t^9, t^3+3t^7]$$

```
> alphaprime:=diff(alpha,t);
```

$$alphaprime := [1, 3t^2]$$

```
> grand1:=innerprod(Falpha,alphaprime);
```

$$grand1 := t^9+3(t^3+3t^7)t^2$$

```
> Lint1:=Int(grand1,t=0..1);
```

$$Lint1 := \int_0^1 t^9+3(t^3+3t^7)t^2\,dt$$

```
> V1:=value(Lint1);
```

$$V1 := \frac{3}{2}$$

For our second path, we must start at $(1,1)$ and end at $(0,0)$ along $y=x$. Note how we reverse the usual parameterization.

```
> beta:=[1-t,1-t];
```

$$\beta := [1-t, 1-t]$$

```
> Fbeta:=F(op(beta));
```

$$Fbeta := [(1-t)^3, 4(1-t)^3]$$

```
> betaprime:=diff(beta,t);
```

$$betaprime := [-1, -1]$$

```
> grand2:=innerprod(Fbeta,betaprime);
```

$$grand2 := 5(-1+t)^3$$

```
> Lint2:=Int(grand2,t=0..1);
```

$$Lint2 := \int_0^1 5(-1+t)^3\, dt$$

```
> V2:=value(Lint2);
```

$$V2 := \frac{-5}{4}$$

```
> Lintanswer:=V1+V2;
```

$$Lintanswer := \frac{1}{4}$$

From this we may conclude that $\oint_C \vec{F} \cdot d\vec{r} = \frac{1}{4}$. Now we will evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ and compare the answers.

```
> Nx:=diff(N(x,y),x);  My:=diff(M(x,y),y);
```

$$Nx := 3x^2 + 3y^2$$
$$My := 3y^2$$

```
> grandGT:=Nx-My;
```

$$grandGT := 3x^2$$

```
> ansGT:=Doubleint(grandGT,y=x^3..x,x=0..1);
```

$$ansGT := \int_0^1 \int_{x^3}^{x} 3x^2\, dy\, dx$$

```
> GTanswer:=value(ansGT);
```

$$GTanswer := \frac{1}{4}$$

And as we expected, the two answers agree.

C3M15 Problems

I. Evaluate the line integrals directly, using pencil and paper.

1. $\int_C (x^2 - y^3)\, dx + (x^2 + y^2)\, dy$, $C : (1,0) \to (0,1)$ on $x^2 + y^2 = 1$.
2. $\int_C (x^2 - y^2)\, dy$, $C : (0,0) \to (1,2)$ on $y = 2x^2$.
3. $\int_C (x + 2y)\, dx + y\, dy$, C counterclockwise on the ellipse $x^2 + 4y^2 = 1$
4. $\int_{(1,0)}^{(0,1)} y(e^{xy} + 1)\, dx + x(e^{xy} + 1)\, dy$

II. Use Green's Theorem to evaluate the line integrals, using pencil and paper.

5. $\int_C (x^2 y + y^3 - \ln x)\, dx + (x \cos y + xy^2)\, dy$, where C is the polygonal path $(1,1) \to (2,1) \to (2,2) \to (1,2) \to (1,1)$
6. $\int_C (\sin y - x^2 y)\, dx + (x \cos y + xy^2)\, dy$, C is the circle $x^2 + y^2 = 1$ counterclockwise.
7. $\int_C \left(e^x \sin y - xy^2\right) dx + \left(e^x \cos y + x^2 y\right) dy$, C is the ellipse $9x^2 + 4y^2 = 36$ counterclockwise.