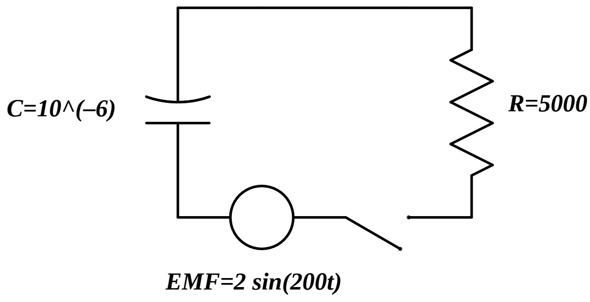
III. Example 1: R-C AC Circuit

An R-C circuit consists of a $2\sin(200t)$ V AC generator connected in series with a 5 kΩ resistor and a 1 μF capacitor. (This problem is from the example on p. 175 of Hambley: **Electrical Engineering—Principles & Applications**, 1997.) Assume current starts to flow and that the capacitor is initially charged to 10^{-6} coulombs when the open switch is closed.



Task: Write down the Initial Value Problem associated with this circuit and solve it for the charge on the capacitor.

Solution: By Kirchhoff's law we have: $E_R + E_C = E$, with $E_R = R \cdot Q'(t)$ and $E_C = Q(t)/C$, translates into the IVP

$$5000\,Q'(t) + \frac{Q(t)}{10^{-6}} = 2\sin(200t)\,, \qquad Q(t) = 10^{-6} \quad \text{at} \quad t = 0$$

1. Divide through by 5000 to get the ODE in standard form

$$Q'(t) + 200\,Q(t) = 0.0004\,\sin(200t)$$

2. Use the coefficient 200 of $Q(t)$ in the standard form of the ODE to compute the *integrating factor*

$$\mu(t) = e^{\int 200\,dt} = e^{200t}$$

3. Multiply both sides of the standard form ODE by the integrating factor

$$e^{200t}\,Q'(t) + 200\,e^{200t}\,Q(t) = 0.0004\,e^{200t}\sin(200t)$$

and use the product rule for derivatives to rewrite the left hand side:

$$\left[e^{200t}\,Q(t)\right]' = 0.0004\,e^{200t}\,\sin(200t)$$

4. Integrate both sides of the preceding equation to get (see, e.g., Example 4 on p.399 of Stewart: **Calculus–Concepts and Contexts**, 2nd ed)

$$\begin{aligned} e^{200t}\, Q(t) &= \int 0.0004\, e^{200t} \sin(200t)\, dt \\ &= 10^{-6}\, e^{200t} \left[-\cos(200t) + \sin(200t)\right] + C \end{aligned}$$

which, upon dividing through by e^{200t} yields the general solution to the ODE

$$Q(t) = 10^{-6} \left[-\cos(200t) + \sin(200t)\right] + C\, e^{-200t}$$

5. From the Initial Condition $Q(0) = 10^{-6}$ we get from this last equation

$$10^{-6} = Q(0) = 10^{-6} \left[-\cos(0) + \sin(0)\right] + C\, e^{0}$$

$$\Longrightarrow 10^{-6} = 10^{-6} \left[-1 + 0\right] + C \cdot 1$$

$$\Longrightarrow 10^{-6} = -10^{-6} + C$$

$$\Longrightarrow C = 2 \times 10^{-6}$$

So the solution to the IVP and the charge on the capacitor is

$$Q(t) = 10^{-6} \left(-\cos(200t) + \sin(200t) + 2e^{-200t}\right)$$

The current $I(t) = Q'(t)$ will then be

$$\begin{aligned} I(t) &= 10^{-6} \left(200 \sin(200t) + 200 \cos(200t) - 400\, e^{-200t}\right) \\ &= 200 \sin(200t) + 200 \cos(200t) - 400 e^{-200t} \text{ microamps} \end{aligned}$$

Graphs of charge and current are shown below.