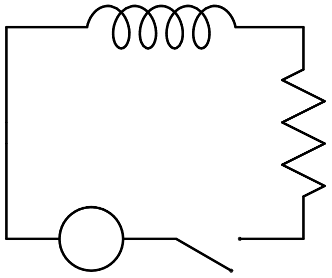
III. Example 2: R-L AC Circuit

Physical characteristics of the circuit: EMF $E(t) = 100\sin(4t)$ connected in series with a 2 henry inductor and a 6 ohm resistor; current flows when the open switch is closed.



Questions:

[a] Describe in words how the current changes over time.

[b] What is the current 1 second after the switch is closed?

[c] At what time does the current equal 8 amps?

[d] What is the largest current achieved and when is it achieved?

Solution of IVP.

By Kirchhoff's laws we have: $E_L + E_R = EMF$ which translates, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, into the following Initial Value Problem (for $t \geq 0$):

$$2\,I'(t) + 6\,I(t) = 100\sin(4t), \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

After looking closely at this ODE, we realize that we cannot use the method of separation of variables because the variables I and t cannot be isolated on separate sides of the equality sign.

A common technique to solve ODEs like this one is to introduce an *integrating factor*.

Outline of solution by *integrating factor*

After dividing both sides of the ODE in

$$2\,I'(t) + 6\,I(t) = 100\sin(4t), \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

by 2, we get the ODE in standard form

$$I' + 3I = 50\sin(4t) \tag{$*$}$$

Since 3 is the coefficient of I in $(*)$, then the integrating factor

$$\mu = e^{\int 3\,dt} = e^{3t}$$

Multiply both side of $(*)$ by μ and integrate to get

$$e^{3t} I = e^{3t}\left[-8\,\cos(4t) + 6\,\sin(4t)\right] + C$$

Divide through by e^{3t} and use the initial condition $I(0) = 0$ in order to get the circuit current

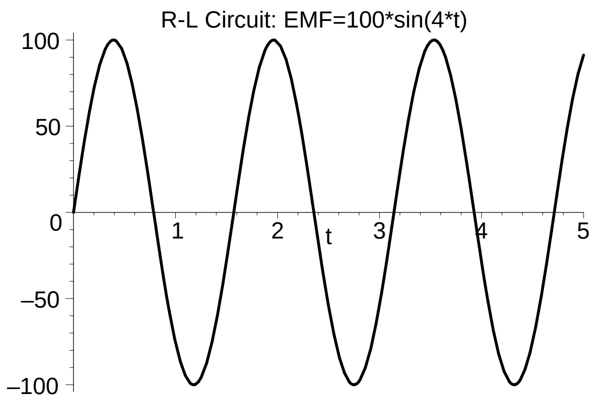
$$I(t) = -8\,\cos(4t) + 6\,\sin(4t) + 8e^{-3t}$$

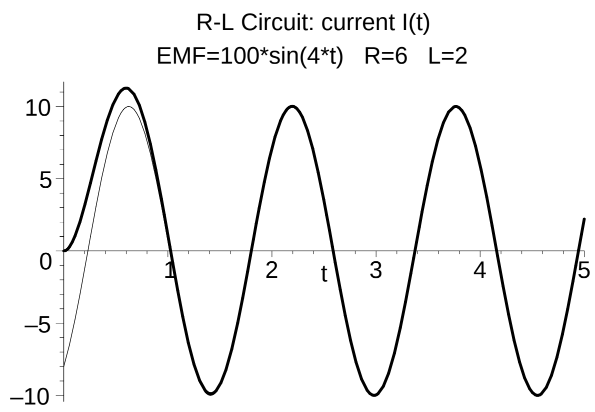
More details for all these steps may be found below, after the Answers.

Answers:

[a] Describe in words how the current changes over time.

The graph below left is that of the EMF $E(t) = 100\sin(4t)$ which oscillates with amplitude 100 and completes one cycle in period $2\pi/4 = \pi/2 \approx 1.57$ seconds. The graph of I shown below right (dark curve) suggests that the current soon after $t = 0$ also appears oscillatory with amplitude 10 and period $\pi/2$.





To establish this last observation, we first see that after a couple of seconds, the "transient term" $8e^{-3t}$ of the current is very nearly 0, at which time

$$I \approx -8\cos(4t) + 6\sin(4t)$$

We can use trigonometry identities to write this as

$$I \approx 10\sin(4t + \phi)$$

where

$$\phi = 2\arctan\frac{-8}{6 + \sqrt{(-8)^2 + (6)^2)}} = 2\arctan(-0.5)\ \text{radians} \approx -53^\circ$$

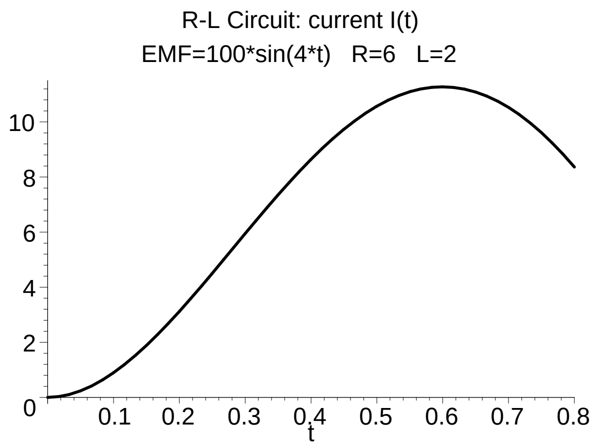
is called the *phase angle*. The initial portion of the graph of $10\sin(4t+\phi)$ is shown as a lighter colored curve in the last graph. So indeed, after the transient term dies off, the current behaves like the EMF: oscillating with amplitude 10, period $\pi/2$, and with a horizontal time shift.

[b] What is the current 1 second after the switch is closed?

$$I(1) = -8\cos(4) + 6\sin(4) + 8e^{-3} \approx 1.0866 \text{ amps}$$

[c] At what time does the current equal 8 amps?

From the oscillatory behavior of $I(t)$ we see that it equals 8 amps infinitely many times. Using a graphing calculator we get an approximation (by tracing the curve or zooming) for the *first time* $I(t) = 8$ at $t \approx 0.374$ seconds. The following plot suggests this answer is correct. Also, we should numerically check that $I(0.374) \approx 7.99$.



[d] What is the largest current achieved and when is it achieved?

Using a graphing calculator, we get the maximum current to be about 11.27 amps at about 0.6 seconds after the switch is closed. See the preceding graph of $I(t)$.

Details of solution by *integrating factor*

After dividing both sides of the ODE in

$$2\,I'(t) + 6\,I(t) = 100\sin(4t), \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

by 2, we get the ODE in standard form

$$I' + 3I = 50\sin(4t) \tag{$*$}$$

and the left hand side of this ODE reminds us of the product rule for derivatives

$$[f(t)g(t)]' = f'(t)g(t) + f(t)g'(t)$$

In fact, if we let $f(t) = I(t)$ and $g(t) = e^{3t}$ then we have

$$\left[Ie^{3t}\right]' = [I]'e^{3t} + I\left[e^{3t}\right]' = I'e^{3t} + 3Ie^{3t}$$

So if we multiply both sides of equation $(*)$ by the *integrating factor* $\mu = e^{\int 3\,dt} = e^{3t}$, we get:

$$\begin{aligned} I'e^{3t} + 3Ie^{3t} &= 50e^{3t}\sin(4t) \\ \left[Ie^{3t}\right]' &= 50e^{3t}\sin(4t) \\ Ie^{3t} &= 50\int e^{3t}\sin(4t)\,dt \end{aligned}$$

when we integrate. The right hand side of the last equation is calculated (see, for example, chapter 5 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed, and in particular Example 4 on p. 399) by the method of *integration by parts*, which gives us

$$Ie^{3t} = 50e^{3t}\left[\frac{-4}{3^2+4^2}\cos(4t) + \frac{3}{3^2+4^2}\sin(4t)\right] + C$$

Multiplying both sides of this last equation by e^{-3t} and simplifying the fractions, we are left with

$$I(t) = -8\cos(4t) + 6\sin(4t) + Ce^{-3t}$$

Using the initial condition $I(0) = 0$ we get

$$\begin{aligned} 0 = I(0) &= -8\cos(0) + 6\sin(0) + Ce^0 \\ \Longrightarrow 0 &= -8 + 0 + C\cdot 1 \\ \Longrightarrow 0 &= -8 + C \\ \Longrightarrow C &= 8 \end{aligned}$$

So for any time t we have the current

$$I(t) = -8\cos(4t) + 6\sin(4t) + 8e^{-3t}$$