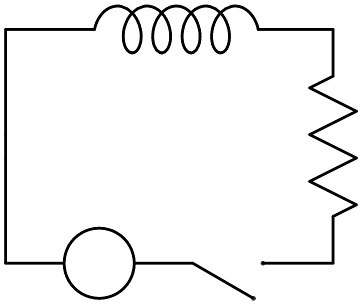
III. Example 2: R-L Circuit with general EMF

Physical characteristics of the circuit: a 4 henry inductor and a 12 ohm resistor connected in series with an unspecified EMF $E(t)$; current flows when the open switch is closed.



Task: Determine the circuit current formula in terms of the unknown EMF function $E(t)$.

Solution:

By Kirchhoff's laws we have: $E_L + E_R = EMF$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the following Initial Value Problem (for $t \geq 0$):

$$4\,I'(t) + 12\,I(t) = E(t), \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

We will use the method of *integrating factors*.

[a] Put the ODE in standard form.

After dividing through by the coefficient of $I'(t)$ the ODE

$$4\,I'(t) + 12\,I(t) = E(t)$$

becomes

$$I'(t) + 3\,I(t) = \frac{1}{4}E(t)$$

[b] Determine the integrating factor μ.

From the general standard form ODE $y' + p(x)\,y = q(x)$ we recognize that

$$I'(t) + 3\,I(t) = \frac{1}{4}E(t)$$

has $p(t) = 3$ and so

$$\mu = e^{\int p(t)\,dt} = e^{\int 3\,dt} = e^{3t}$$

[c] Multiply the standard form ODE by the integrating factor.

Standard form:

$$I' + 3\,I = \frac{1}{4}E(t)$$

Integrating factor:

$$\mu = e^{3t}$$

Product:

$$e^{3t}I' + 3\,e^{3t}I = \frac{1}{4}e^{3t}E(t)$$

[d] Use the product rule for derivatives to simplify the preceding equation so that $[\mu\, I]'$ is on one side of it.

By the product rule

$$e^{3t}I' + 3\,e^{3t}I = \left[e^{3t}\,I\right]'$$

So the preceding ODE

$$e^{3t}I' + 3\,e^{3t}I = \frac{1}{4}e^{3t}E(t)$$

is equivalent to

$$\left[e^{3t}\,I\right]' = \frac{1}{4}e^{3t}E(t)$$

[e] Integrate both sides of the preceding equation with respect to t.

Integrating both sides of

$$\left[e^{3t}\,I\right]' = \frac{1}{4}e^{3t}E(t)$$

yields

$$e^{3t}I = \frac{1}{4}\int e^{3t}E(t)\,dt$$

For our purposes, it is better to write this in terms of a *definite* integral:

$$e^{3t}I = \frac{1}{4}\int_0^t e^{3u}E(u)\,du + C$$

(This is part of "The Fundamental Theorem of Calculus, Part 1"; see, for example, p 382 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed.)

[f] Solve the preceding equation for the solution I.

From

$$e^{3t}I = \frac{1}{4}\int_0^t e^{3u}E(u)\,du + C$$

we get

$$I = \frac{1}{4}e^{-3t}\int_0^t e^{3u}E(u)\,du + C\,e^{-3t}$$

Using the initial condition $I(0) = 0$ we solve for C:

$$0 = I(0) = \frac{1}{4} e^{0} \int_0^0 e^{3u} E(u)\, du + C\, e^{0}$$

$$\Longrightarrow 0 = \frac{1}{4} \cdot 1 \cdot 0 + C \cdot 1$$

$$\Longrightarrow 0 = 0 + C$$

$$\Longrightarrow C = 0$$

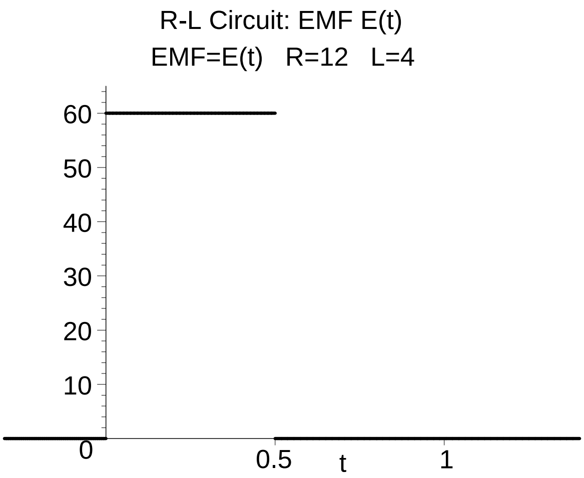
So the current can be evaluated at any time t by

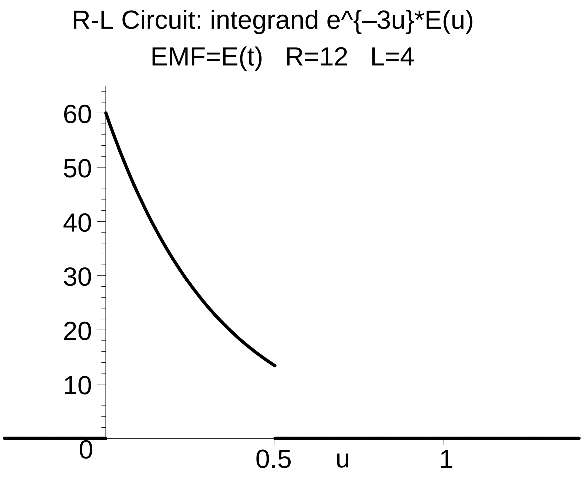
$$I(t) = \frac{1}{4} e^{-3t} \int_0^t e^{3u} E(u)\, du \qquad (*)$$

Application: Let's look at the case when the EMF in the circuit is a 60 volt battery that is turned on at the same time the circuit switch is closed at $t = 0$ and then turned off $1/2$ second later. We represent $E(t)$ as a *piecewise defined function*

$$E(t) = \begin{cases} 60, & \text{if } 0 \le t < 1/2 \\ 0, & \text{if } 1/2 \le t \end{cases}$$

The following two plots show both the EMF (lower left) and the integrand of the integral in equation $(*)$ (lower right).





Our formula in equation $(*)$ is based on computing the area under the upper right curve between $u = 0$ and $u = t$. Since that curve "changes" at $t = 1/2$, we have two cases to consider.

Case 1. We have from equation $(*)$ for $0 \le t < 1/2$:

$$\begin{aligned}
I(t) &= \frac{1}{4}\,e^{-3t}\int_0^t e^{3u}E(u)\,du \\
&= \frac{1}{4}\,e^{-3t}\int_0^t e^{3u}\,60\,du \\
&= \frac{1}{4}\cdot 60\cdot e^{-3t}\int_0^t e^{3u}\,du \\
&= 15\,e^{-3t}\cdot\frac{1}{3}e^{3u}\Big]_{u=0}^{u=t} \\
&= 5\,e^{-3t}\left(e^{3\cdot t}-e^{3\cdot 0}\right) \\
&= 5\,e^{-3t}\left(e^{3t}-e^{0}\right) \\
&= 5\left(e^{-3t}\cdot e^{3t}-e^{-3t}\cdot 1\right) \\
&= 5\left(1-e^{-3t}\right)
\end{aligned}$$

And if the battery is not turned off, this would be the current for all $t \ge 0$.

Case 2. For $1/2 \le t$:

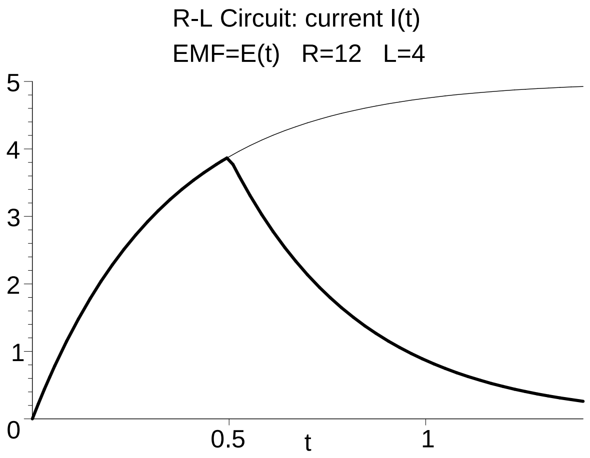
$$\begin{aligned}
I(t) &= \frac{1}{4}e^{-3t}\int_0^t e^{3u}E(u)\,du \\
&= \frac{1}{4}e^{-3t}\left(\int_0^{1/2} e^{3u}\,60\,du + \int_{1/2}^t e^{3u}\,0\,du\right) \\
&= 5\,e^{-3t}\left(e^{3/2}-e^{0}\right)+0 \\
&= 5\left(e^{3/2}-1\right)e^{-3t}
\end{aligned}$$

using steps similar to Case 1.

Putting the results of these two cases together:

$$I(t) = \begin{cases} 5\left(1-e^{-3t}\right), & \text{if} \quad 0 \le t < 1/2 \\ 5\left(e^{3/2}-1\right)e^{-3t}, & \text{if} \quad 1/2 \le t \end{cases}$$

This current shown in the next graph as the thicker curve with the thinner curve representing the current if the battery is not switched off at $t = 1/2$.



Can you see how the current changes when no longer driven by the 60 volt EMF?