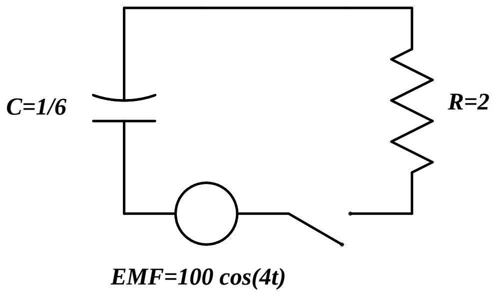
## III. Practice Problem 1: R-C AC Circuit

**Task:** Work on solving for the charge and current for the given circuit; indicated links give (partial) solutions.




An R-C circuit consists of a $100\cos(4t)$ volt AC generator connected in series with a 2 henry inductor and a $1/6$ farad capacitor; there is no initial charge on the capacitor and no current flows until the switch is closed. The associated Initial Value Problem is

$$2\,Q'(t) + \frac{Q(t)}{1/6} = 100\cos(4t), \quad Q(0) = 0$$

### Method of Integrating Factor:

[a] Put the ODE in standard form.

[b] Determine the integrating factor $\mu$.

[c] Multiply the standard form ODE by the integrating factor.

[d] Use the product rule for derivatives to simplify the preceding equation so that $[\mu\,Q]'$ is on one side of it.

[e] Integrate both sides of the preceding equation with respect to $t$.

[f] Solve the preceding equation for the general solution $Q$.

[g] Use the initial condition to determine the charge $Q(t)$ for this circuit.

[h] Determine the current $I(t)$ for this circuit.

[a] Put the ODE in standard form.

$$2\,Q'(t) + \frac{Q(t)}{1/6} = 100\cos(4t)$$

becomes

$$Q'(t) + 3\,Q(t) = 50\cos(4t)$$

[b] Determine the integrating factor $\mu$.

From the general standard form ODE $y' + p(x)\,y = q(x)$ we recognize that

$$Q'(t) + 3\,Q(t) = 50\cos(4t)$$

has $p(t) = 3$ and so

$$\mu = e^{\int p(t)\,dt} = e^{\int 3\,dt} = e^{3t}$$

[c] Multiply the standard form ODE by the integrating factor.

Standard form:

$$Q'(t) + 3\,Q(t) = 50\cos(4t)$$

Integrating factor:

$$\mu = e^{3t}$$

Product:

$$e^{3t}Q' + 3\,e^{3t}Q = 50\,e^{3t}\cos(4t)$$

[d] Use the product rule for derivatives to simplify the preceding equation so that $[\mu\, Q]\,'$ is on one side of it.

By the product rule

$$e^{3t}Q' + 3\,e^{3t}Q = \left[e^{3t}\,Q\right]\,'$$

So the preceding ODE

$$e^{3t}Q' + 3\,e^{3t}Q = 50\,e^{3t}\cos(4t)$$

is equivalent to

$$\left[e^{3t}\,Q\right]\,' = 50\,e^{3t}\cos(4t)$$

[e] Integrate both sides of the preceding equation with respect to $t$.

Integrating both sides of

$$\left[e^{3t}\,Q\right]' = 50\,e^{3t}\cos(4t)$$

yields

$$e^{3t}Q = \int 50\,e^{3t}\cos(4t)\,dt$$

[f] Solve the preceding equation for the general solution $Q$.

Using integration by parts on the right hand side of

$$e^{3t}Q = \int 50\, e^{3t} \cos(4t)\, dt$$

we get

$$e^{3t}Q = 50\, e^{3t} \left[ \frac{3}{3^2+4^2} \cos(4t) + \frac{4}{3^2+4^2} \sin(4t) \right] + C$$

which, after simplifying the numbers and dividing both sides by $e^{3t}$, becomes

$$Q(t) = 6\cos(4t) + 8\sin(4t) + C\, e^{-3t}$$

[g] Use the initial condition to determine the charge $Q(t)$ for this circuit.

From the general solution

$$Q(t) = 6\cos(4t) + 8\sin(4t) + C\,e^{-3t}$$

in the preceding part we have at $t = 0$

$$Q(0) = 6\cos(0) + 8\sin(0) + C\,e^{0} = 6 \cdot 1 + 8 \cdot 0 + C = 6 + C$$

whereas by the initial condition from the IVP we have

$$Q(0) = 0$$

So these two equations together imply that

$$6 + C = 0 \qquad \text{or} \qquad C = -6$$

Putting this $C$ into the general solution yields the circuit charge

$$Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$$

[h] Determine the current $I(t)$ for this circuit.

Since current is the time derivative of charge $I(t) = Q'(t)$ we get from the preceding part

$$I(t) = \left[6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}\right]'$$

or

$$I(t) = -24\sin(4t) + 32\sin(4t) + 18e^{-3t}$$

for $t > 0$.