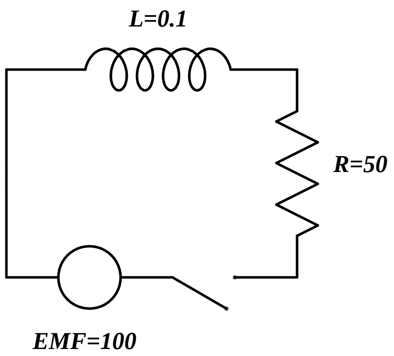
III. Practice Problem 2: R-L DC Circuit

Task: Work on solving for the current for the given circuit; indicated links give (partial) solutions.



An R-L circuit consists of a 100 volt DC battery connected in series with a 0.1 henry inductor and a 50 ohm resistor; current flows when the open switch is closed. (This is the example on pp. 168–9 of Hambley: **Electrical Engineering—Principles & Applications**). The associated Initial Value Problem is

$$0.1\,I'(t) + 50\,I(t) = 100, \quad I(0) = 0$$

Method of Separation of Variables:

[a] Put the ODE in differential form.

[b] Use algebra to separate variables.

[c] Integrate both sides of the preceding equation to get a general solution to the ODE.

[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.

[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the current $I(t)$ for this circuit.

[f] Rework the solution when there is an initial current of 5 amps in this circuit at time $t = 0$.

[a] Put the ODE in differential form.

The ODE

$$0.1\, I'(t) + 50\, I(t) = 100$$

becomes

$$I'(t) + 500\, I(t) = 1000$$

or

$$\frac{dI}{dt} = 1000 - 500\, I$$

which, after multiplying through by dt, becomes

$$dI = (1000 - 500\, I)\, dt$$

[b] Use algebra to separate variables.

The differential form

$$dI = (1000 - 500\,I)\,dt$$

can be written

$$\frac{dI}{1000 - 500\,I} = dt$$

so that all terms involving I are on one side of the equation and all terms involving t on the other side.

[c] Integrate both sides of the preceding equation to get a general solution to the ODE.

Integrating both sides of

$$\frac{dI}{1000 - 500\,I} = dt$$

gives

$$\int \frac{dI}{1000 - 500\,I} = \int dt$$

We use the substitution

$$\begin{aligned} x &= 1000 - 500\,I \\ \text{to get} \qquad \frac{dx}{dI} &= -500 \\ \text{or} \qquad dI &= -0.002\,dx \end{aligned}$$

so that

$$\begin{aligned} \int \frac{1}{1000 - 500\,I}\,dI &= \int \frac{1}{x}\,(-0.002)\,dx \\ &= -0.002 \int \frac{1}{x}\,dx \\ &= -0.002 \ln|x| + C \\ &= -0.002 \ln|1000 - 500\,I| + C \end{aligned}$$

Hence the general solution to the ODE in implicit form is

$$-0.002 \ln|1000 - 500\,I| = t + C$$

[d] Use the initial condition to compute the value of the arbitrary constant and determine the implicit solution to the IVP.

Since $I = 0$ when $t = 0$, we put $t = 0$ into

$$-0.002 \ln |1000 - 500\, I| = t + C$$

to get

$$-.002 \ln |1000 - 0| = 0 + C \Longrightarrow C = -0.002 \ln 1000$$

This leads us to the solution to the IVP in implicit form:

$$-0.002 \ln |1000 - 500\, I| = t - 0.002 \ln 1000$$

or, multiplying through by -500,

$$\ln |1000 - 500\, I| = \ln 1000 - 500t$$

[e] Use the rules of logs and exponentials in order to solve the implicit solution to the ODE for the current $I(t)$ for this circuit.

Take the antilogarithm of both sides of the solution to the IVP

$$\ln|1000 - 500\,I| = \ln 1000 - 500t$$

to get

$$e^{\ln|1000-500\,I|} = e^{\ln 1000 - 500t}$$

which, when simplified, yields

$$|1000 - 500\,I| = 1000\,e^{-500t}$$

or

$$1000 - 500\,I = \pm 1000\,e^{-500t}$$

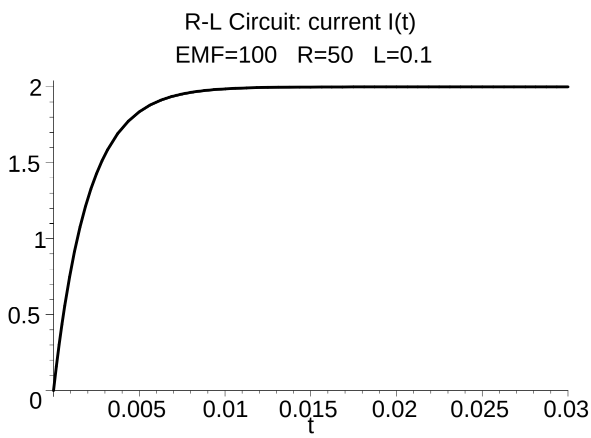
Since $I = 0$ when $t = 0$, the sign must be $+$, allowing us to solve for I:

$$1000 - 500\,I = +1000\,e^{-500t}$$

or

$$I(t) = 2 - 2\,e^{-500t}$$

whose graph is plotted below.



[f] Rework the solution when there is an initial current of 5 amps in this circuit at time $t = 0$.

Start with the general solution

$$-0.002 \ln |1000 - 500\, I| = t + C$$

from part [c] and substitute $I = 5$ and $t = 0$:

$$-0.002 \ln |1000 - 500(5)| = t + C$$

which yields

$$C = -0.002 \ln(1500)$$

Hence

$$-0.002 \ln |1000 - 500(5)| = t - 0.002 \ln(1500)$$

which we can solve as before.

Multiply by -500:

$$\ln |1000 - 500\, I| = -500t + \ln(1500)$$

Take antilogs:

$$|1000 - 500\, I| = e^{-500t + \ln(1500)}$$

Simplify exponentials:

$$e^{-500t + \ln(1000)} = e^{-500t} \times e^{\ln(1500)} = e^{-500t} \times 1500$$

to arrive at:

$$|1000 - 500\, I| = 1500\, e^{-500t}$$

or

$$1000 - 500\, I = \pm 1500\, e^{-500t}$$

Now $I = 5$ when $t = 0$ means

$$1000 - 500(5) = \pm 1500\, e^{0}$$

or

$$-1500 = \pm 1500$$

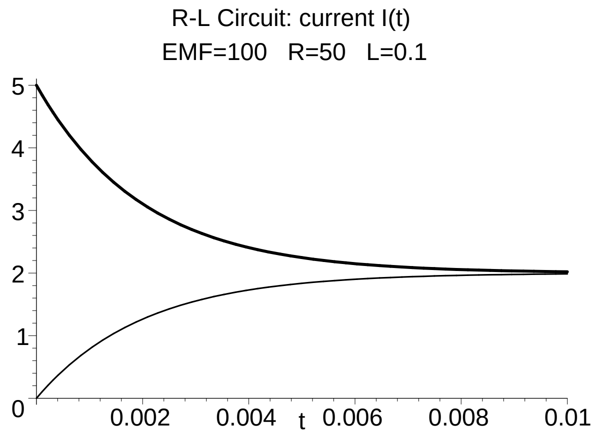
So the sign must be $-$ and the solution to the IVP is

$$1000 - 500\, I = -1500\, e^{-500t}$$

or

$$I(t) = 2 + 3\, e^{-500t}$$

Current is plotted below for $I(0) = 0$ (thinner curve) and $I(0) = 5$ (thicker curve).


R-L Circuit: current I(t)
EMF=100 R=50 L=0.1
5
4
3
2
1
0
0.002
0.004
t
0.006
0.008
0.01