II. Example 1: Solving First Order Linear ODE by Integrating Factor

Task: Solve the initial value problem (IVP)

$$3\,y' + 15\,y = 24\,e^{3x}\,, \qquad y = -2 \quad \text{when} \qquad x = 0 \tag{$*$}$$

Solution: Method of Integrating Factor

1. Divide through by 3 to get the ODE in $(*)$ into *standard form* with 1 as the coefficient of y'

$$y' + 5\,y = 8\,e^{3x}$$

2. Define the *integrating factor* from the coefficient of y in the preceding equation

$$\mu(x) = e^{\int 5\,dx} = e^{5x}$$

where we may take the arbitrary constant of integration to be 0.

3. Multiply both sides of the standard form ODE

$$y' + 5\,y = 8\,e^{3x}$$

by the integrating factor e^{5x} to get

$$e^{5x}\,y' + 5\,e^{5x}\,y = 8\,e^{8x}$$

which can be written (using the product rule for derivatives) as

$$\left[e^{5x}\,y\right]' = 8\,e^{8x}$$

4. Integrate both sides of this last equation to get

$$e^{5x}\,y = \int 8\,e^{8x}\,dx$$

or

$$e^{5x}\,y = e^{8x} + C$$

and divide through by e^{5x} to get the solution to the ODE in $(*)$ as

$$y = e^{3x} + C\,e^{-5x}$$

using the law of exponents $e^{-5x} \times e^{8x} = e^{3x}$.

5. To satisfy the intial condition $y(0) = -2$ in $(*)$ we put $y = -2$ and $x = 0$ into

$$y = e^{3x} + C\,e^{-5x}$$

resulting in

$$\begin{aligned} -2 &= e^{3\cdot 0} + C\,e^{5\cdot 0} \\ \Longrightarrow -2 &= 1 + C\cdot 1 \\ \Longrightarrow -2 &= 1 + C \\ \Longrightarrow -3 &= C \end{aligned}$$

So the solution to the IVP $(*)$ is

$$y = e^{3x} - 3\,e^{-5x}$$