II. Example 1: First Order Separable ODEs

Task: Solve the initial value problem (IVP)

$$y' = \frac{3e^{3x}}{y+5}\,, \qquad y = 1 \quad \text{when} \quad x = 0 \tag{$*$}$$

Solution: Method of Separation of Variables

1. Multiply both sides of the ODE in $(*)$ by $y+5$ to get it in the form

$$(y+5)\,y' = 3e^{3x}$$

2. Replace y' by (the Leibniz) differential form $\dfrac{dy}{dx}$

$$(y+5)\,\frac{dy}{dx} = 3e^{3x}$$

3. Multiply both sides of this last ODE by dx:

$$(y+5)\,dy = 3e^{3x}\,dx$$

thus "separating variables": *each side of the equation now has one and only one variable present.*

4. Integrate both sides

$$\int (y+5)\,dy = \int 3e^{3x}\,dx$$

5. Evaluate both indefinite integrals and put all arbitrary constants on the right hand side

$$\frac{1}{2}y^2 + 5y = e^{3x} + C$$

6. To solve for y, multiply both sides of the preceding equation by 2

$$y^2 + 10y = 2e^{3x} + C$$

and complete the square (or use the quadratic equation) to get

$$(y+5)^2 - 25 = 2e^{3x} + C$$

or

$$(y+5)^2 = 2e^{3x} + C$$

where 25 gets absorbed into the arbitrary constant C. Taking the square root of both sides of the preceding equation

$$y + 5 = \sqrt{2e^{3x} + C}$$

which leads to

$$y = \sqrt{2e^{3x} + C} - 5$$

as the general solution to the ODE in $(*)$. To satisfy the initial condition $y(0) = 1$ in $(*)$ we need to have

$$\begin{aligned} \sqrt{2e^{3\cdot 0} + C} - 5 &= 1 \\ \implies \qquad \sqrt{2 \cdot 1 + C} &= 6 \\ \implies \qquad 2 + C &= 36 \\ \implies \qquad C &= 34 \end{aligned}$$

So the solution to the IVP $(*)$ is

$$y = \sqrt{2e^{3x} + 34} - 5$$