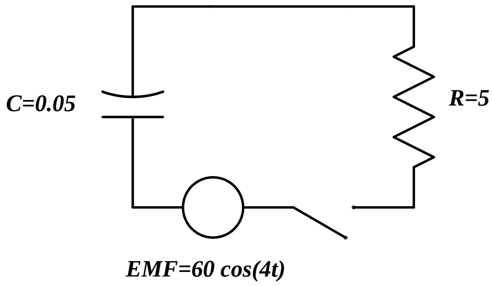
II. Example 2: Solve R-C AC Circuit by Integrating Factor

An R-C circuit consists of a $60\cos(4t)$ volt AC generator connected in series with a 5 ohm resistor and a 0.05 farad capacitor. Assume current starts to flow and that there is no charge on the capacitor when the open switch is closed.



Task: Write down the Initial Value Problem associated with this circuit and solve it for the charge on the capacitor.

Solution: By Kirchhoff's law: $E_R + E_C = E$, with $E_R = R \cdot Q'(t)$ and $E_C = Q(t)/C$, translates into the IVP

$$5\,Q'(t) + \frac{Q(t)}{0.05} = 60\cos(4t)\,, \qquad Q(t) = 0 \quad \text{at} \quad t = 0$$

1. Divide through by 5 to get the ODE in standard form

$$Q'(t) + 4\,Q(t) = 12\cos(4t)$$

2. Use the coefficient 4 of $Q(t)$ in the standard form of the ODE to compute the *integrating factor*

$$\mu(t) = e^{\int 4\,dt} = e^{4t}$$

3. Multiply both sides of the standard form ODE by the integrating factor

$$e^{4t}\,Q'(t) + 4\,e^{4t}\,Q(t) = 12\,e^{4t}\,\cos(4t)$$

and use the product rule for derivatives to rewrite the left hand side:

$$\left[e^{4t}\,Q(t)\right]' = 12\,e^{4t}\,\cos(4t)$$

4. Integrate both sides of the preceding equation to get (see, e.g., Example 4 on p.399 of Stewart: **Calculus–Concepts and Contexts**, 2nd ed)

$$\begin{aligned} e^{4t}\,Q(t) &= \int 12\,e^{4t}\,\cos(4t)\,dt \\ &= 1.5\,e^{4t}\,(\cos(4t) + \sin(4t)) + C \end{aligned}$$

which, upon dividing through by e^{4t}, yields the general solution to the ODE

$$Q(t) = 1.5\,(\cos(4t) + \sin(4t)) + C\,e^{-4t}$$

5. From the Initial Condition $Q(0) = 0$ we get from this last equation

$$0 = Q(0) = 1.5\,(\cos(0) + \sin(0)) + C\,e^{0}$$

$$\Longrightarrow 0 = 1.5\,(1 + 0) + C \cdot 1$$

$$\Longrightarrow 0 = 1.5 + C$$

$$\Longrightarrow C = -1.5$$

So the solution to the IVP and the charge on the capacitor is

$$Q(t) = 1.5\left(\cos(4t) + \sin(4t) - e^{-4t}\right)$$