II. Solving First Order Separable ODEs

An ordinary differential equation of the form

$$y' = \frac{p(x)}{q(y)}$$

is called a *first order separable ODE* where $p(x)$ is a function of only the variable x and $q(y) \neq 0$ is a function of only the variable y.

Solution: Method of Separation of Variables

1. Multiply both sides of the ODE by $q(y)$ to get it in the form

$$q(y)\, y' = p(x)$$

2. Replace y' by (the Leibniz) differential form $\dfrac{dy}{dx}$

$$q(y)\, \frac{dy}{dx} = p(x)$$

3. Multiply both sides of this last ODE by dx:

$$q(y)\, dy = p(x)\, dx$$

thus "separating variables": each side of the equation now has one and only one variable present.

4. Integrate both sides (an allowable operation because of the Chair Rule for derivatives)

$$\int q(y)\, dy = \int p(x)\, dx$$

5. If both integrals can be evaluated, then we have an "implicit" solution to the original ODE in the form

$$Q(y) = P(x) + C$$

after collecting the integrals' arbitrary constants on one side as C.

6. You might be able to solve that last equation for y in terms of x

$$y = F(x)$$

or perhaps x in terms of y

$$x = G(y)$$

in order to get an “explcit” solution to the original ODE.

Special case: *Linear first order ODE with constant coefficients*

To solve

$$a\,y' + b\,y = c$$

where $a \neq 0\,,\ b \neq 0$, and c are constants.

1. Divide through by the number a to get an ODE of the form

$$y' + p\,y = q$$

where numbers $p = b/a$ and $q = c/a$.

2. Subtract $p\,y$ from both sides and write y' in differential notation $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = q - p\,y$$

3. Divide both sides of the ODE by $q - p\,y$, multiply both sides by dx

$$\frac{dy}{q - p\,y} = dx$$

and then integrate

$$\int \frac{dy}{q - p\,y} = \int dx$$

4. Use the *method of substitution* with $u = q - p\,y$ so that $du = -p\,dy$ or $dy = (-1/p)du$ to evaluate the left hand side of the preceding equation

$$\begin{aligned} \int \frac{dy}{q - p\,y} &= \int \frac{(-1/p)du}{u} \\ &= \frac{-1}{p} \ln|u| + C_1 \\ &= \frac{-1}{p} \ln|q - p\,y| + C_1 \end{aligned}$$

where C_1 is an arbitrary constant.

5. Putting parts 3. and 4. together

$$\frac{-1}{p} \ln|q - p\,y| + C_1 = x + C_2$$

or, after combining the two arbitrary constants on the right side

$$\ln|q - p\,y| = -p\,x + C$$

which, after taking the antilogarithm of both sides, yields

$$\begin{aligned} |q - p\,y| &= e^{-p\,x + C} \\ &= e^{-p\,x} e^{C} \\ &= A\,e^{-p\,x} \end{aligned}$$

where $A = e^C$ is an arbitrary positive constant. Then after eliminating the absolute values we are left with

$$q - p\,y = \pm A\,e^{-p\,x}$$

from which we can solve for y

$$y = \frac{q}{p} - \frac{\pm A}{p} e^{-p\,x}$$

We replace $\frac{\pm A}{p}$ by C to stand for an arbitrary constant. And looking back we see that since $q = c/a$ and $p = b/a$, the solution to the ODE may be written

$$y = \frac{c}{b} + C\,e^{-\frac{b}{a}\,x}$$