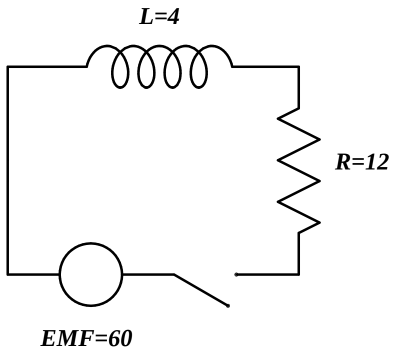
I. Example 1: R-L DC Circuit

Note: a circuit with constant EMF is called a *direct current circuit* or *DC circuit*.

Questions:

[a] Use Kirchhoff's law to write the Initial Value Problem — ODE and initial condition(s) — for the simple circuit consisting of a 60 volt DC battery connected in series with a 4 henry inductor and a 12 ohm resistor. Current flows when the open switch is closed. (Note: This is Example #2 on p. 515 and #3 on p. 524 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed.)



[b] Verify that $I(t) = 5\left(1 - e^{-3t}\right)$, $t \geq 0$ is the solution to the IVP in part [a].

[c] Graph $I(t)$. What is the asymptotic limit of $I(t)$ as $t \to \infty$? This is called the *steady-state* current and we will label it I_∞.

[d] At what time t does the current $I(t)$ reach 99% of its steady state value?

Answers:

[a] By Kirchhoff's law we have that $E_L + E_R = E$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the Initial Value Problem (for $t \geq 0$)

$$4\, I'(t) + 12\, I(t) = 60, \qquad I(t) = 0 \quad \text{at} \quad t = 0 \tag{$*$}$$

[b] If $I(t) = 5\left(1 - e^{-3t}\right)$ then its derivative

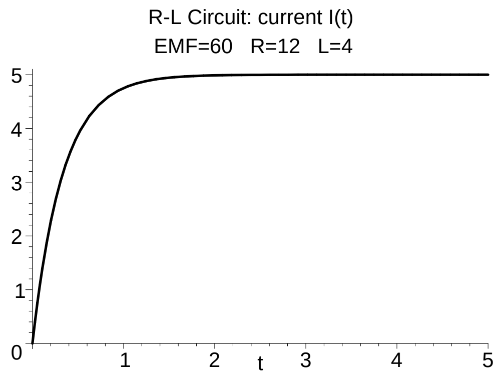
$$I'(t) = 5\left(0 - (-3)e^{-3t}\right) = 5\left(3\, e^{-3t}\right) = 15e^{-3t}$$

and so the left hand side of the ODE in $(*)$ above becomes

$$\begin{aligned} 4\, I'(t) + 12\, I(t) &= 4\left(15e^{-3t}\right) + 12\left(5\left(1 - e^{-3t}\right)\right) \\ &= 60\, e^{-3t} + 60 - 60\, e^{-3t} \\ &= 60 \end{aligned}$$

Hence $I(t)$ satisfies the ODE of $(*)$. Also, $I(0) = 5\left(1 - e^{0}\right) = 5(1-1) = 0$ and $I(t)$ also satisfies the IC of $(*)$.

[c] As $t \to \infty$ we have $I(t) = 5\left(1 - e^{-3t}\right) \to 5\,(1-0) = 5$ as indicated in the following graph of $I(t)$. The steady-state current $I_\infty = 5$ amps.



We can see that as $t \to \infty$ and the graph of $I(t)$ levels off toward the steady-state value I_∞, then the slope of the graph $I'(t) \to 0$. If we take the limit $\lim_{t\to\infty}$ of both sides of the ODE in $(*)$ we get

$$\lim_{t\to\infty}\left(4\,I'(t) + 12\,I(t)\right) = \lim_{t\to\infty}(60)$$

which yields $4 \cdot 0 + 12\,I_\infty = 60$ or $I_\infty = 60/12 = 5$.

Remark. An R-L circuit with constant DC voltage E has steady-state current $I_\infty = \dfrac{E}{R}$.

[d] We use algebra to solve $I(t) = 0.99 I_\infty$. In this circuit, this is the same as

$$5\left(1 - e^{-3t}\right) = 0.99(5)$$

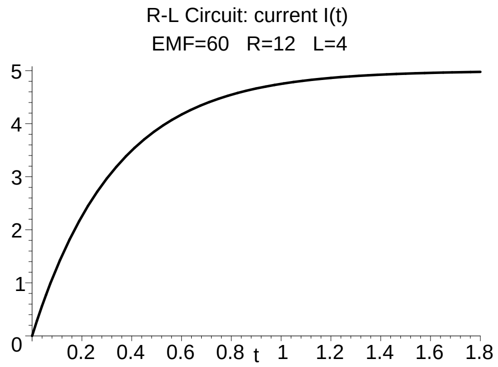
which, after canceling the fives, leads to

$$\begin{aligned} 1 - e^{-3t} &= 0.99 \\ \Longrightarrow e^{-3t} &= 0.01 \\ \Longrightarrow -3t &= \ln(0.01) \\ \Longrightarrow t &= -\frac{\ln(0.01)}{3} \approx 1.535 \end{aligned}$$

This answer could also have been approximated by plotting $I(t)$ on a graphing calculator and zooming or tracing the curve to see where $I(t)$ achieves the value 4.95 amps.

Remark. In R-L DC circuits, a *time unit* τ is defined by $\tau = \dfrac{L}{R}$. After 5 time units, the current will be at a little more than 99% of its steady-state: $I(5\tau) \approx 0.9933\,I_\infty$.

In this example, $\tau = L/R = 4/12 = 1/3$ and $5\tau = 5/3 \approx 1.67$ seconds. We see in the next graph of $I(t) = 5\left(1 - e^{-3t}\right)$ that the current is indeed very near $I_\infty = 5$ when t is past 1.67 seconds.



Algebraically,

$$I\left(5/3\right) = 5\left(1 - e^{-3(5/3)}\right) = 5\left(1 - e^{-5}\right) \approx 5(0.99326) \approx 4.9663$$