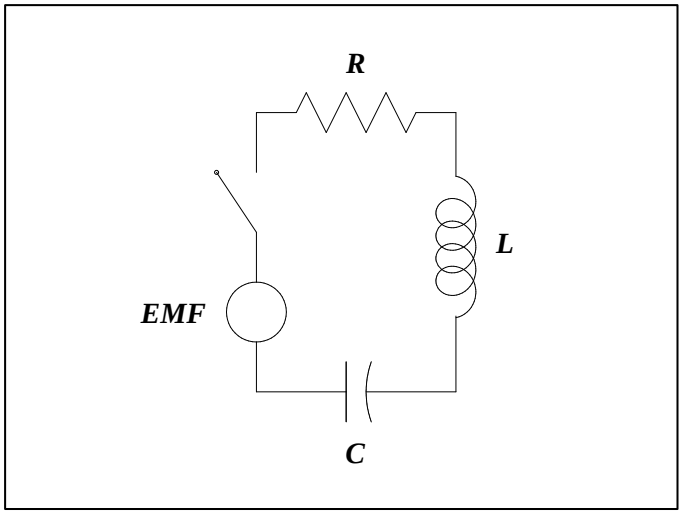
ODEs and Electric Circuits

Introduction: Modeling Electric Circuits (Kirchhoff's Law)

One of the topics that you will study in your Physics and Electrical Engineering courses at the Naval Academy is that of electric circuits. Consider the single-loop series circuit below.



[a] The device labeled EMF, which stands for *electromotive force*, can be a battery or generator. The EMF supplies current to the circuit.

[b] The device labeled C is a *capacitor* which stores charge.

[c] The device labeled R is a *resistor* which opposes the flow of current.

[d] The device labeled L is an *inductor* which opposes change in the flow of the current.

The last device in the circuit is a switch, shown open. When the switch is closed, a current $I(t)$ runs through the circuit which changes the charge $Q(t)$ at the capacitor. The following table summarizes what the measurements are, their symbols, and units.

Circuit Measurements		
measurement	**symbol**	**units**
time	t	seconds (sec)
charge	$Q(t)$	Coulombs
current	$I(t)$	Amperes (amps)
electromotive force (EMF)	$E(t)$	Volts (V)

When the current is flowing, there is a corresponding voltage drop across each of the devices. The next table summarizes the device symbols, units, and voltage drops.

Circuit Devices			
device name	**symbol**	**units**	**voltage drop**
capacitor	C	Farads (F)	$E_C = \frac{1}{C} \cdot Q$
resistor	R	Ohms (Ω)	$E_R = R \cdot I$
inductor	L	Henries (H)	$E_L = L \cdot \frac{dI}{dt}$

At any time t, the current $I(t)$ and the charge $Q(t)$ are related by two important rules:

1. The current equals the rate of change of the charge (1 Amp = 1 Coulomb/sec):

$$I(t) = \frac{dQ(t)}{dt}$$

2. **Kirchhoff's Law:** (the electrical equivalent of Newton's second law of motion)

The sum of the voltage drops around the closed circuit is equal to the voltage supplied: $E_L + E_R + E_C = E$

or

$$L\,\frac{dI}{dt} + R\,I + \frac{1}{C}\,Q = E$$

Since $I = \dfrac{dQ}{dt}$, we get

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{1}{C}\,Q = E$$

which is a second order ODE for the charge $Q(t)$.

There are two special cases that we can solve with the methods introduced in this course.

R-C Circuits: If there is no inductor, we can investigate a first order ODE for the charge Q

$$R\,\frac{dQ}{dt} + \frac{1}{C}\,Q = E$$

R-L Circuits: If there is no capacitor, we can investigate a first order ODE for the current I

$$L\,\frac{dI}{dt} + R\,I = E$$

Example I. **R-L Circuit with constant EMF**

Note: a circuit with constant EMF is called a "direct current circuit" or *DC circuit* for short.

[a] Use Kirchhoff's law to write the Initial Value Problem — ODE and initial condition(s) — for the simple circuit consisting of a 100 volt DC generator connected in series with a 2 henry inductor and a 6 ohm resistor. Current flows when the open switch is closed.
[b] Verify that $I(t) = \frac{50}{3}\left(1 - e^{-3t}\right)$, $t \geq 0$ is the solution to your IVP in part [a].
[c] Graph $I(t)$. What is the asymptotic limit of $I(t)$ as $t \to \infty$?

Solution.
[a] By Kirchhoff's law we have that $E_L + E_R = E$ which translates into the Initial Value Problem (for $t \geq 0$)

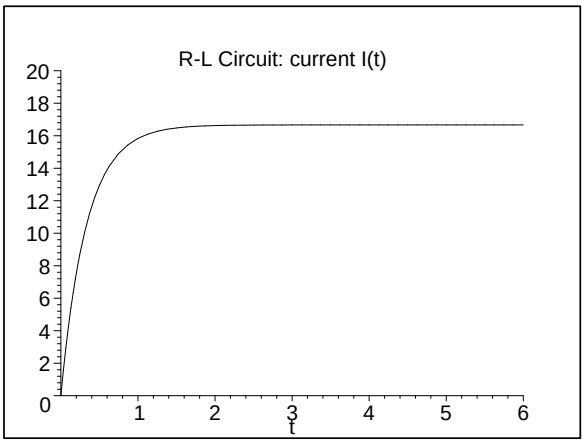
$$2\,I'(t) + 6\,I(t) = 100, \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

[b] If $I(t) = \frac{50}{3}\left(1 - e^{-3t}\right)$ then

$$\begin{aligned} 2\,I'(t) + 6\,I(t) &= 2\left(\frac{50}{3}\left(1 - e^{-3t}\right)\right)' + 6\left(\frac{50}{3}\left(1 - e^{-3t}\right)\right) \\ &= 2\left(\frac{50}{3}\left(3\,e^{-3t}\right)\right) + 100\left(1 - e^{-3t}\right) \\ &= 100\,e^{-3t} + 100 - 100\,e^{-3t} \\ &= 100 \end{aligned}$$

Also, $I(0) = \frac{50}{3}\left(1 - e^{0}\right) = \frac{50}{3}(1 - 1) = 0$.

[c] As $t \to \infty$ we have $I(t) = \frac{50}{3}\left(1 - e^{-3t}\right) \to \frac{50}{3}\,(1 - 0) = \frac{50}{3}$ as indicated in the following graph of $I(t)$.



Example II. **R-C Circuit with oscillating EMF**

Note: a circuit with oscillating or periodic EMF is called an "alternating current circuit" or *AC circuit* for short.

[a] Use Kirchhoff's law to write the Initial value Problem — ODE and initial condition(s) — for the simple circuit consisting of a $100\cos(4t)$ volt AC generator connected in series with a 2 ohm resistor and a $\frac{1}{6}$ farad capacitor. The charge on the capacitor is 0 when the open switch is closed.
[b] Verify that $Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$ is the solution to your IVP in part [a].
[c] Find $I(t)$, the current at time t, then graph both charge $Q(t)$ and current $I(t)$..

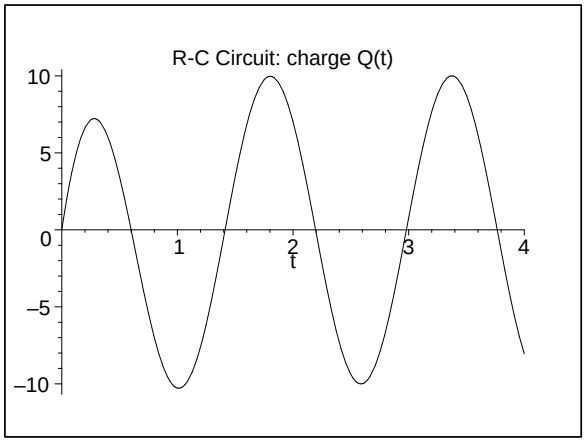
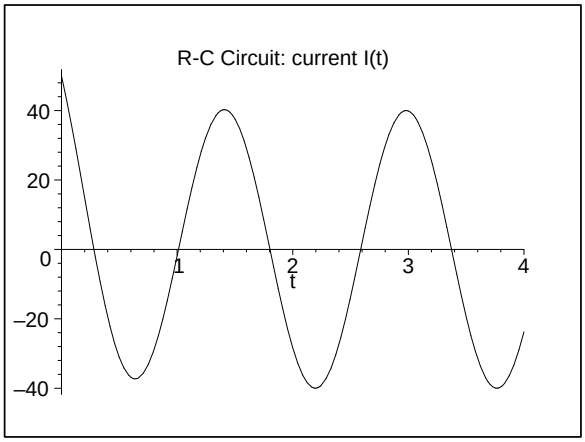
Solution.
[a] By Kirchhoff's law we have that $E_R + E_C = E$ so

$$2\,I(t) + \frac{1}{1/6}Q(t) = 100\cos(4t) \qquad \text{or} \qquad Q'(t) + 3\,Q(t) = 50\cos(4t)$$

and $Q(0) = 0$.

[b] If $Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$ then

$$\begin{aligned} Q'(t) + 3\,Q(t) &= \left(6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}\right)' + 3\left(6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}\right) \\ &= \left(-24\sin(4t) + 32\cos(4t) + 18\,e^{-3t}\right) + \left(18\cos(4t) + 24\sin(4t) - 18\,e^{-3t}\right) \\ &= 50\cos(4t) \end{aligned}$$

and $Q(0) = 6\cos(0) + 8\sin(0) - 6\,e^0 = 6 + 0 - 6 = 0$.

[c] $I(t) = Q'(t) = -24\sin(4t) + 32\cos(4t) + 18e^{-3t}$ for $t > 0$. The graphs of $Q(t)$ and $I(t)$ are below.