Solving First Order Linear ODE by Integrating Factor

A **first order linear ODE** is any differential equation of the form

$$a(x)\,y' + b(x)\,y = c(x)$$

where $a(x) \neq 0$, $b(x)$, and $c(x)$ are functions of x. To solve this ODE:

1. Divide through by $a(x)$ to get a first order ODE in *standard form*

$$y' + p(x)\,y = q(x)$$

where $p(x) = b(x)/a(x)$ and $q(x) = c(x)/a(x)$.

2. Define the *integrating factor*

$$\mu(x) = e^{\int p(x)\,dx}$$

3. Multiply both sides of the ODE by the integrating factor

$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,q(x)$$

and notice that the ODE can then be written

$$[\mu(x)\,y]' = \mu(x)\,q(x)$$

because by the product rule for derivatives

$$[\mu(x)\,y]' = \mu(x)\,y' + \mu(x)'y$$

and by the chain rule for derivatives

$$\begin{aligned}\mu(x)' &= \frac{d}{dx}\left[e^{\int p(x)\,dx}\right] \\ &= e^{\int p(x)\,dx}\,\frac{d}{dx}\left[\int p(x)\,dx\right] \\ &= \mu(x)\,p(x)\end{aligned}$$

4. Integrate both sides of

$$[\mu(x)\,y]' = \mu(x)\,q(x)$$

to get

$$\mu(x)\,y = \int \mu(x)\,q(x)\,dx$$

and divide through by $\mu(x)$ to arrive at the solution to the ODE

$$y = \frac{1}{\mu(x)}\int \mu(x)\,\frac{c(x)}{a(x)}\,dx$$

where the integrating factor $\mu(x) = e^{\int \frac{b(x)}{a(x)}\,dx}$

Example III. Use the integrating factor method to find the general solution for

$$x\,y' + y = \cos(x)$$

1. Divide by x to get the standard form ODE

$$y' + \frac{1}{x}\,y = \frac{\cos(x)}{x}$$

2. The integrating factor is

$$\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

3. Multiply both sides of the standard form ODE by x to get $x\,y' + 1\,y = \cos(x)$ and rewrite this as $[x\,y]\,' = \cos(x)$.

4. Integrate both sides

$$\int [x\,y]\,'dx = \int \cos(x)\,dx$$

so $x\,y = \sin(x) + C$, and then divide by x to get the ODE solution

$$y = \frac{\sin(x)}{x} + \frac{C}{x}$$

Example IV. Solve Example II by the integrating factor method

1. $Q'(t) + 3\,Q(t) = 50\cos(4t), \quad Q(0) = 0$
2. $\mu = e^{\int 3\,dt} = e^{3t}$
3. $e^{3t}Q' + 3\,e^{3t}Q = 50\,e^{3t}\cos(4t)$ or

$$\left[e^{3t}\,Q\right]\,' = 50\,e^{3t}\cos(4t)$$

4.

$$e^{3t}Q = \int 50\,e^{3t}\cos(4t)\,dt = 50\,e^{3t}\left[\frac{3}{3^2+4^2}\cos(4t) + \frac{4}{3^2+4^2}\sin(4t)\right] + C$$

(using integration by parts) which, upon dividing through by e^{3t}, leads to the solution of the ODE

$$Q(t) = 6\cos(4t) + 8\sin(4t) + C\,e^{-3t}$$

For the initial condition: $Q(0) = 6\cos(0) + 8\sin(0) + C\,e^0 = 6 + C$ will equal 0 if $C = -6$. So the solution to the IVP is

$$Q(t) = 6\cos(4t) + 8\sin(4t) - 6\,e^{-3t}$$