1. Vectors and vector operations—mechanical

The purpose of this section is to provide you with the basic computational tools for working with vectors. These tools allow you to manipulate vectors just as a hammer, a screwdriver, a saw, etc. allow you to manipulate nails, screws, wood, etc. The actual use of these tools will be introduced in Section 2.

After finishing Section 1, you should be able to:

- Translate between bracket and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation for vectors.
- Classify an expression as vector-valued, scalar-values, or undefined.
- Calculate the sum and difference of two vectors.
- Calculate the product of a scalar, and the quotient of a vector by a scalar.
- Calculate the magnitude of a vector.
- Calculate the dot product of two vectors.
- Calculate the cross product of two vectors.
- Perform arbitrarily complex vector calculations by combining the above operations.

1.1. Vectors.

For the time being, we shall consider a vector as simply a list of three numbers such as $\langle 1, 0, -10\rangle$ or $\langle x, y, z\rangle$. The numbers in the list are called the *components* of the vector. Several operations, such as addition and multiplication (even more than one kind of multiplication) are defined for combining and manipulating vectors in the way we are used to handling ordinary numbers.

Vectors, and all the vector operations described in this section have very useful geometric interpretations; but we will postpone consideration of these for the time being. Our first goal is to master the mechanics of calculating the various vector operations.

A *scalar* is the term we use to describe an ordinary real number, as opposed to a vector. Examples of scalars: -3, 0, 100, π.

In printed text, it is common practice to use boldface letters such as $\mathbf{a}$, $\mathbf{b}$ to denote vectors and italic letters such as a, b to denote scalars. It is convenient to make the distinction as clear as possible since the operations for vectors and scalars are quite different. In handwriting one usually writes a small arrow over a letter representing a vector: $\vec{a}$, $\vec{b}$.

We use the symbol $\mathbf{0}$ to denote the zero vector $\langle 0, 0, 0\rangle$.

1.2. Vector addition and subtraction.
Input: two vectors Output: a vector
Formula:

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

$$\langle a_1, a_2, a_3\rangle - \langle b_1, b_2, b_3\rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle.$$

Example:

$$\langle 3, 2, -7\rangle + \langle 2, -8, 4\rangle = \langle 3 + 2, 2 - 8, -7 + 4\rangle = \langle 5, -6, -3\rangle.$$

$$\langle 3, 2, -7\rangle - \langle 2, -8, 4\rangle = \langle 3 - 2, 2 - (-8), -7 - 4\rangle = \langle 1, 10, -11\rangle.$$

Your turn: Let $\mathbf{a} = \langle 4, 2, -1\rangle$ and $\mathbf{b} = \langle 3, 0, 3\rangle$. Find $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. Do it now. (*Answer:* $\langle 7, 2, 2\rangle$ and $\langle 1, 2, -4\rangle$)

Algebraic properties of vector addition and subtraction include the following (most will be familiar):

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

$$(\mathbf{a} - \mathbf{b}) - \mathbf{c} = \mathbf{a} - (\mathbf{b} + \mathbf{c}).$$

$$(\mathbf{a} + \mathbf{b}) - \mathbf{c} = \mathbf{a} + (\mathbf{b} - \mathbf{c}).$$

$$(\mathbf{a} - \mathbf{b}) + \mathbf{c} = \mathbf{a} - (\mathbf{b} - \mathbf{c}).$$

These properties (as well as those for the other vector operations) are easily verified by using the defining formulas.

1.3. Vector-scalar addition and subtraction.

There is *no* way to add or subtract a vector and a scalar.

1.4. Scalar multiplication, scalar division.
Input: a vector and a scalar Output: a vector
Formula:

$$s\langle a_1, a_2, a_3\rangle = \langle a_1, a_2, a_3\rangle s = \langle sa_1, sa_2, sa_3\rangle.$$

Example:

$$-8\langle 3, 2, -7\rangle = \langle -24, -16, 56\rangle.$$

A vector $\mathbf{a}$ can also be divided by a nonzero scalar s by multiplying $\mathbf{a}$ times $1/s$.
Example:

$$\langle 2, -5, -1\rangle/3 = \langle 2/3, -5/3, -1/3\rangle.$$

Your turn: Let $\mathbf{a} = \langle 4, 2, -1\rangle$. Find $5\mathbf{a}$ and $\mathbf{a}/5$. Do it now. (*Answer:* $\langle 20, 10, -5\rangle$ and $\langle .8, .4, -.2\rangle$)

Algebraic properties of the scalar product include the following (most will be familiar):

$$s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}.$$

$$(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}.$$

$$(st)\mathbf{a} = s(t\mathbf{a}).$$

1.5. Alternative vector notation.

Certain particular vectors are given special symbols:

$$\mathbf{i} = \langle 1, 0, 0\rangle;\ \mathbf{j} = \langle 0, 1, 0\rangle;\ \mathbf{k} = \langle 0, 0, 1\rangle.$$

Any vector can be (and often is) expressed in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ using the operations of vector addition, vector subtraction, and scalar multiplication.
Example:

$$\langle 9, -3, 0\rangle = 9\mathbf{i} - 3\mathbf{j} + 0\mathbf{k} = 9\mathbf{i} - 3\mathbf{j}.$$

Get used to using both notations for vectors.

1.6. Magnitude of a vector.
Input: a vector Output: a nonnegative scalar
Formula:

$$||\langle a_1, a_2, a_3\rangle|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example:

$$||3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}|| = \sqrt{3^2 + 2^2 + (-7)^2} = \sqrt{62}.$$

Your turn: Let $\mathbf{a} = \langle 4, 2, -1\rangle$. Find $||\mathbf{a}||$. Do it now. (*Answer:* $\sqrt{21}$)

Other names for the magnitude of a vector are the *norm* or the *absolute value*. Understand these when you encounter them.

Algebraic properties of the magnitude include the following:

$$||\mathbf{a} + \mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||.$$

$$||s\mathbf{a}|| = |s|\ ||\mathbf{a}||.$$

1.7. Dot product.
Input: two vectors Output: a scalar
Formula:

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3.$$

Example:

$$(4\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} + 9\mathbf{k}) = (4)(2) + (-5)(1) + (1)(9) = 12.$$

Your turn: Let $\mathbf{a} = \langle 4, 2, -1\rangle$ and $\mathbf{b} = \langle 2, -2, 4\rangle$. Find $\mathbf{a} \cdot \mathbf{b}$. Do it now. (*Answer:* 0)

Other names for the dot product are *scalar product* (not to be confused with scalar multiple) or the *inner product*. Understand these when you encounter them.

Algebraic properties of the dot product (most should seem familiar):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}.$$

$$\mathbf{c} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{b}.$$

$$\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2.$$

$$s(\mathbf{a} \cdot \mathbf{b}) = (s\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (s\mathbf{b}).$$

1.8. Cross product.
Input: two vectors Output: a vector
Formula:

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle.$$

If you know how to calculate a 3×3 determinant, you can also remember the cross product as:

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Another method, popular with students, is what we'll call the *tic-tac-toe* method. This requires that you memorize the nine multiplication facts $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$. First write the components of the two vectors at the top and left side of a grid diagram as shown.

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$			
$a_2\mathbf{j}$			
$a_3\mathbf{k}$			

Then fill in the contents of the table by cross-product. Take care to get the signs right.

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$	$\mathbf{0}$	$a_1b_2\mathbf{k}$	$-a_1b_3\mathbf{j}$
$a_2\mathbf{j}$	$-a_2b_1\mathbf{k}$	$\mathbf{0}$	$a_2b_3\mathbf{i}$
$a_3\mathbf{k}$	$a_3b_1\mathbf{j}$	$-a_3b_2\mathbf{i}$	$\mathbf{0}$

Finally, collect terms:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \end{aligned}$$

Example:

$$\begin{aligned}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 9\mathbf{k}) &= \langle -5 \cdot 9 - 1 \cdot 1, 1 \cdot 2 - 4 \cdot 9, 4 \cdot 1 - (-5) \cdot 2 \rangle \\ &= \langle -46, -34, 14 \rangle.\end{aligned}$$

Using the tic-tac-toe method, we would have:

	$2\mathbf{i}$	$\mathbf{j}$	$9\mathbf{k}$
$4\mathbf{i}$	$\mathbf{0}$	$4\mathbf{k}$	$-36\mathbf{j}$
$-5\mathbf{j}$	$10\mathbf{k}$	$\mathbf{0}$	$-45\mathbf{i}$
$\mathbf{k}$	$2\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

$$\begin{aligned}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 9\mathbf{k}) &= 4\mathbf{k} - 36\mathbf{j} + 10\mathbf{k} - 45\mathbf{i} + 2\mathbf{j} - \mathbf{i} \\ &= -46\mathbf{i} - 34\mathbf{j} + 14\mathbf{k}.\end{aligned}$$

Your turn: Let $\mathbf{a} = \langle 4, 2, -1 \rangle$ and $\mathbf{b} = \langle 2, -2, 4 \rangle$. Find $\mathbf{a} \times \mathbf{b}$. Do it now. (*Answer:* $6\mathbf{i} - 18\mathbf{j} - 12\mathbf{k}$)

Another name for the cross product is *vector product*. Understand this when you encounter it.

Algebraic properties of the cross product (watch out: most are *un*familiar):

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) (\text{usually}).$$

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

1.9. Conclusion

These are all the vector operations we are going to study. Any combination of vectors and scalars not covered by the foregoing rules (such as **ab**, you should consider undefined.

Know the following terms: vector, component, scalar, scalar multiplication, magnitude, norm, absolute value, dot product, inner product, scalar product, cross product, vector product.

Memorize the following:

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle.$$

$$\langle a_1, a_2, a_3\rangle - \langle b_1, b_2, b_3\rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle.$$

$$s\langle a_1, a_2, a_3\rangle = \langle a_1, a_2, a_3\rangle s = \langle sa_1, sa_2, sa_3\rangle.$$

$$||\langle a_1, a_2, a_3\rangle|| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

$$\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3.$$

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle;$$

or

$$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

1.10. Exercises.

1-6: Convert to notation $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$:

1. $\langle 3, -1, 0\rangle$
2. $\langle 0, 1, 0\rangle$
3. $\langle -1, 2, 7\rangle$
4. $\langle 0, 0, 0\rangle$
5. $\langle 10, 0, 1\rangle$
6. $\langle -4, 2, -2\rangle$

7-12: Convert to notation $\langle a, b, c\rangle$:

7. $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
8. $\mathbf{k}$
9. $4\mathbf{i} + 6\mathbf{k}$
10. $\mathbf{j} + 3\mathbf{i}$
11. $\mathbf{k} - \mathbf{i}$
12. $5\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$

Let $\mathbf{a} = \langle 2, -2, 0\rangle$, $\mathbf{b} = 4\mathbf{i} - 9\mathbf{j} + \mathbf{k}$, $\mathbf{c} = \langle 3+t, 13t, 2-t\rangle$, and $\mathbf{r} = \langle x, y, z\rangle$. Find the following, if they exist:

13. $\mathbf{a} + \langle -1, 2, 7\rangle$
14. $\mathbf{a} - \mathbf{b}$
15. $\mathbf{c} + \mathbf{r} + \mathbf{b}$
16. $5 - \langle 3, 0, 1\rangle$
17. $\mathbf{b} + \mathbf{c}$
18. $\mathbf{r} + \langle 4, 0, 0\rangle$
19. $7 + \mathbf{a}$
20. $\langle 2, 4, -6\rangle + \langle 1, 7, 2\rangle$
21. $9\mathbf{c}$
22. $4\mathbf{a}$
23. $6\mathbf{r}$
24. $0\mathbf{c}$
25. $3\mathbf{a} - 2\mathbf{b}$
26. $\mathbf{a} - \mathbf{c} + 10$
27. $\mathbf{a} + 2\mathbf{b} - 4\mathbf{r}$
28. $5\mathbf{a} + 3 + 2$
29. $(4 - 5)\mathbf{b}$
30. $(3 + 10)(\mathbf{a} + \mathbf{c})$
31. $||\mathbf{a}||$
32. $||\langle 5, 12, -13\rangle||$
33. $||2\mathbf{a} - 3\mathbf{b}||$
34. $||3\mathbf{r} + \mathbf{b}||$
35. $\mathbf{a} \cdot \mathbf{b}$
36. $\mathbf{b} \cdot \mathbf{r}$
37. $\mathbf{c} \cdot \langle -2, 1, 4\rangle$
38. $\langle -2, 1, 1\rangle \cdot \mathbf{j}$
39. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$
40. $\mathbf{a} \cdot (\mathbf{b} - \langle 1, 1, -4\rangle)$
41. $\mathbf{i} \cdot \mathbf{j}$
42. $\mathbf{r} \cdot \mathbf{r} - ||\mathbf{r}||^2$
43. $\mathbf{a}/||\mathbf{a}||$
44. $||\mathbf{r}||(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - 4\mathbf{b})$
45. $\mathbf{r}/||\mathbf{r}||$
46. $\mathbf{a} \times \mathbf{b}$
47. $\mathbf{b} \times \mathbf{c}$
48. $\langle 1, 3, 2\rangle \times \mathbf{b}$
49. $\mathbf{i} \times \mathbf{j}$
50. $\mathbf{j} \times \mathbf{k}$
51. $\mathbf{k} \times \mathbf{i}$
52. $\mathbf{i} \times \mathbf{k}$
53. $(\mathbf{a} \times \mathbf{b}) \cdot (2\mathbf{i} - \mathbf{j})$
54. $(\mathbf{a} \times \mathbf{b}) \times \langle 4, -1, 0\rangle$
55. $\mathbf{a} \times (\mathbf{b} \times \langle 4, -1, 0\rangle)$
56. $(\mathbf{a} \cdot \mathbf{b})\langle 4, -1, 0\rangle - (\mathbf{a} \cdot \langle 4, -1, 0\rangle)\mathbf{b}$
57. $||\mathbf{b} \times \mathbf{a}||$
58. $\mathbf{a} \times \mathbf{a}$
59. $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
60. $\mathbf{a} \times (\mathbf{b} + \mathbf{r})$
61. $(\mathbf{a} - 2\mathbf{i} + 3\mathbf{b}) \times (4\mathbf{i} + \mathbf{k})(||\mathbf{b}||)$
62. $\mathbf{ab}$
63. $\mathbf{a}/\mathbf{b}$

64-68: Fill in the other half of the identity.

64. $||\mathbf{a} + \mathbf{b}|| \leq$
65. $||\mathbf{a} \times \mathbf{a}|| =$
66. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =$
67. $\mathbf{a} \cdot \mathbf{a} =$
68. $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} =$

1.11. Answers.

1. $3\mathbf{i} - \mathbf{j}$
2. $\mathbf{j}$
3. $-\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$
4. $\mathbf{0}$
5. $10\mathbf{i} + \mathbf{k}$
6. $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
7. $\langle 1, 3, -1\rangle$
8. $\langle 0, 0, 1\rangle$
9. $\langle 4, 0, 6\rangle$
10. $\langle 3, 1, 0\rangle$
11. $\langle -1, 0, 1\rangle$
12. $\langle 5, 2, -10\rangle$
13 $\langle 1, 0, 7\rangle$
14. $\langle -2, 7, -1\rangle$
15. $\langle 7 + t + x, 13t - 9 + y, 3 - t + z\rangle$
16. undefined.
17. $\langle 7 + t, 13t - 9, 3 - t\rangle$
18. $\langle x + 4, y, z\rangle$
19. undefined.
20. $\langle 3, 11, -4\rangle$
21. $\langle 27 + 9t, 117t, 18 - 9t\rangle$
22. $\langle 8, -8, 0\rangle$
23. $\langle 6x, 6y, 6z\rangle$
24. $\langle 0, 0, 0\rangle$
25. $\langle -2, 12, -2\rangle$
26. undefined.
27. $\langle 10 - 4x, -20 - 4y, 2 - 4z\rangle$
28. undefined.
29. $\langle -4, 9, -1\rangle$
30. $\langle 65 + 13t, 169t - 26, 26 - 13t\rangle$
31. $\sqrt{8}$
32. $13\sqrt{2}$
33. $\sqrt{602}$
34. $\sqrt{(3x + 4)^2 + (3y - 9)^2 + (3z + 1)^2}$
35. 26
36. $4x - 9y + z$
37. $2 + 7t$
38. 1
39. -90
40. 26
41. 0
42. 0
43. $\langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$
44. $-462\sqrt{x^2 + y^2 + z^2}$
45. $\langle x/\sqrt{x^2 + y^2 + z^2}, y/\sqrt{x^2 + y^2 + z^2}, z/\sqrt{x^2 + y^2 + z^2}\rangle$
46. $\langle -2, -2, -10\rangle$
47. $\langle -18 - 4t, 5t - 5, 61t + 27\rangle$
48. $\langle 21, 7, -21\rangle$
49. $\mathbf{k}$
50. $\mathbf{i}$
51. $\mathbf{j}$
52. $-\mathbf{j}$
53. -2
54. $\langle -10, -40, 10\rangle$
55. $\langle -64, -64, 10\rangle$
56. $\langle 64, 64, -10\rangle$
57. $6\sqrt{3}$
58. 0
59. undefined.
60. $\langle -2 - 2z, -2 - 2z, -10 + 2x + 2y\rangle$
61. $\langle -203\sqrt{2}, 0, 812\sqrt{2}\rangle$
62. undefined.
63. undefined.

2. Vectors and vector operations—geometric

We now interpret geometrically the vectors and vector operations that we practiced in the last section. We shall consider the vectors as points in three-dimensional space, and the various vector operations as various geometric operations on these points.

In the previous section you acquired the basic tools—the vector operations—for working with vectors. In this section, you will learn the basic functions of these tools: for example, just as the basic functions of a hammer are hammering nails and pulling nails, the basic functions of the dot product are finding the angle between vectors, determining whether vectors are perpendicular, and projecting one vector onto another.

Once you learn the use of each tool, you can use them in combination to solve a wide variety of three-dimensional geometrical problems.

Vectors are also widely used in physics and engineering to model various physical concepts.

After finishing Section 2, you should be able to:

- Convert two-dimensional vectors back and forth between Cartesian and polar form.
- Explain in geometric terms the effects of the following operations: vector addition, vector subtraction, scalar multiplication, magnitude, dot product, cross product.
- Use the scalar product to determine whether two vectors are parallel or to find a vector parallel to a given vector.
- Use the cross product to determine whether two vectors are parallel.
- Determine whether two vectors are perpendicular.
- Find the angle between two given vectors.
- Be able to calculate the scalar projection of one vector onto another and describe geometrically what this means.
- Be able to calculate the vector projection of one vector onto another and describe geometrically what this means.
- Explain what a unit vector is.
- Find a vector in the same direction as a given vector with a given magnitude.
- Find a vector running from one point to another.
- Find the area of a parallelogram formed by two vectors.
- Find a vector perpendicular to two given vectors.
- Find the volume of a parallelepiped formed by three vectors.

- Know what a “parallelepiped” is, and how to spell it.
- Determine whether three vectors are coplanar.

Learning to perform these tasks reliably and without hesitation is an essential foundation for the next section.

2.1. Vectors.

Just as we can locate any point in the plane by a pair of coordinates (x, y), we can locate any point in three-dimensional space by a triple of coordinates (x, y, z). Recall that a vector is also represented as a a list of three numbers. Henceforth we shall consider a vector as corresponding to a point in three dimensional space (whose three coordinates are the three components of the vector).

Usually a vector $\langle a_1, a_2, a_3\rangle$ is visualized as an arrow: the arrow running from the origin $(0, 0, 0)$ to the point (a_1, a_2, a_3) (Figure 2.1). The vector $\langle a_1, a_2, a_3\rangle$ is called the *position vector* of the point (a_1, a_2, a_3). We denote the position vector of the point $A = (a_1, a_2, a_3)$ by $\mathbf{V}(A)$:

$$\mathbf{V}(a_1, a_2, a_3) = \langle a_1, a_2, a_3\rangle.$$

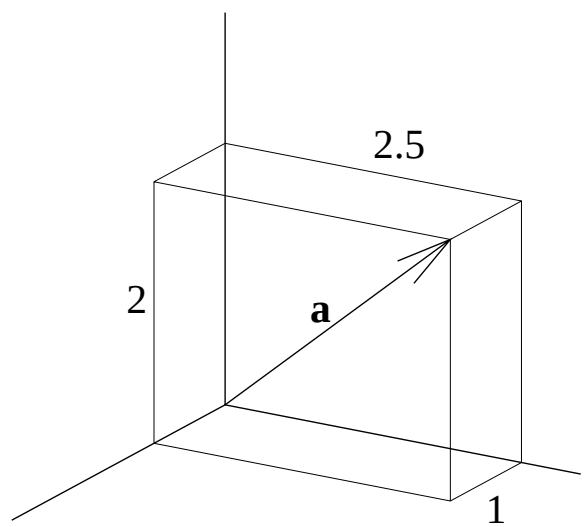

Figure 2.1: The vector $\mathbf{a} = \langle 1, 2.5, 2\rangle$.

A vector has both *magnitude* and *direction*. (The exception is the vector $\mathbf{0}$, which, starting and ending at the same point, has no direction.) Often it is convenient to describe a vector in terms of its direction and magnitude, rather than by the three components.

Location is *not* a distinguishing feature of vectors. We can slide the arrow to any convenient spot; so long as we change neither the length nor the direction of the arrow, it represents the same vector (Figure 2.2).

Our three special vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ become three arrows of length 1 at right angles to each other (Figure 2.3).

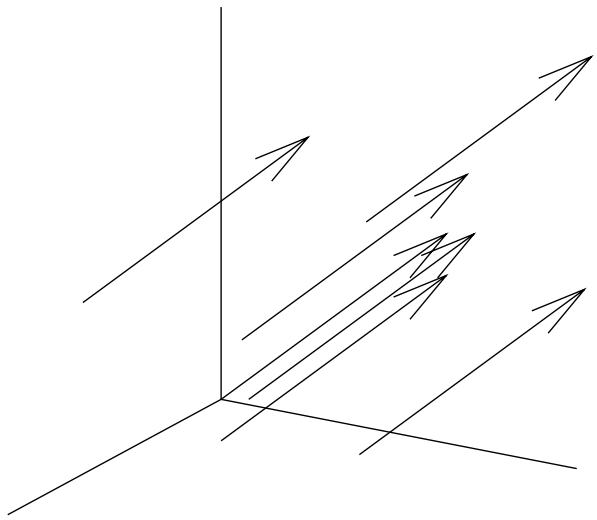

Figure 2.2: The vector **a** drawn in various locations.

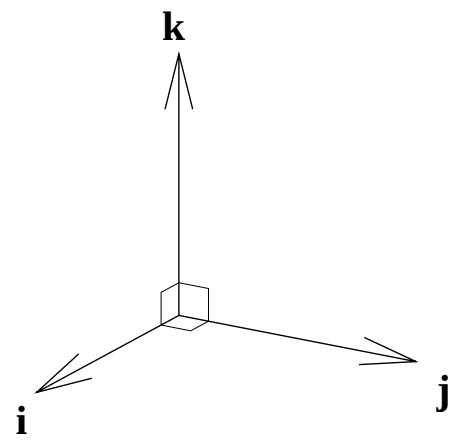

Figure 2.3: The vectors **i**, **j**, and **k**.

2.2. Vector addition and subtraction.

Vectors can be added or subtracted geometrically as well as algebraically (as in the last section). Both methods yield the same result.

To add vectors **a** and **b**, slide the vectors so that the tail of **b** coincides with the head of **a**. The sum $\mathbf{a}+\mathbf{b}$ is then the vector running from the tail of **a** to the head of **b** (Figure 2.4).

Several vectors can be added at once by placing them head-to-tail in sequence (Figure 2.5).

To subtract a vector **b** from a vector **a**, slide the vectors so that the two tails coincide. The difference $\mathbf{a}-\mathbf{b}$ is then the vector running from the head of **b** to the head of **a** (Figure 2.6). Take care not to get this backwards.

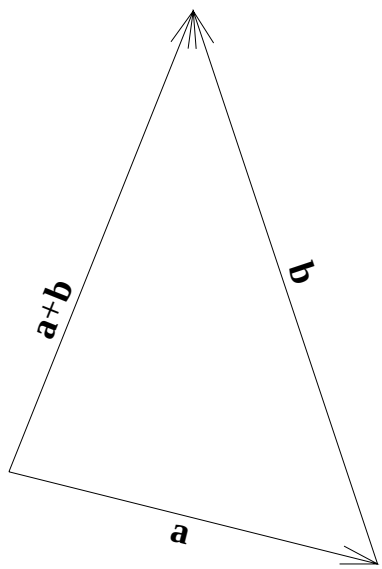

Figure 2.4: Adding vectors **a** and **b**.

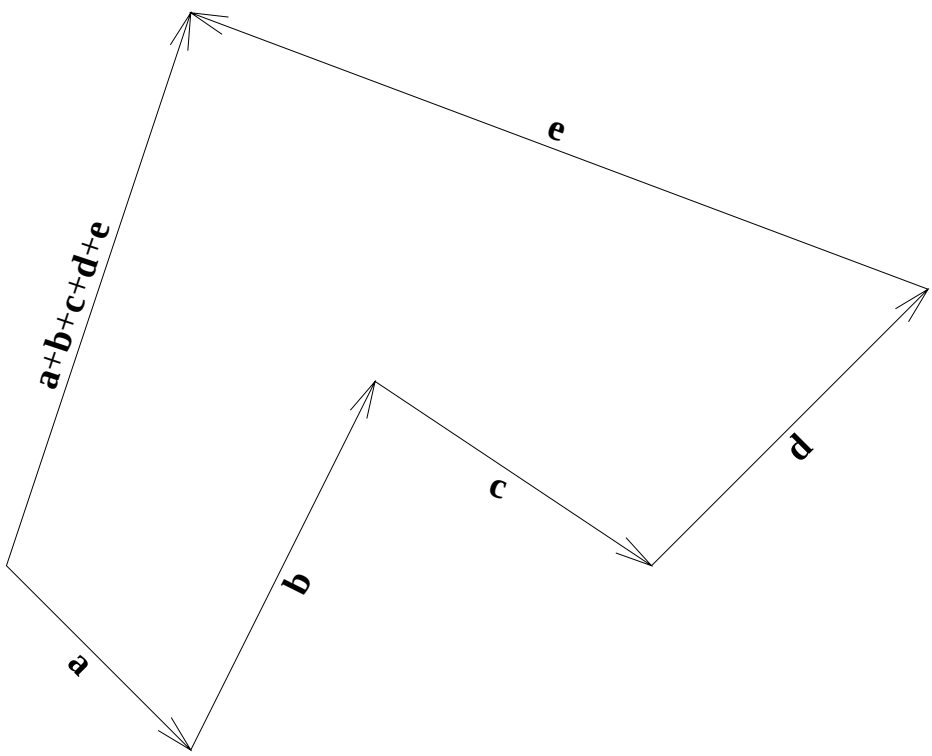

Figure 2.5: Adding vectors **a**, **b**, **c**, **d**, and **e**.

Often we want to find a vector which runs from a point A to a point B. The answer is $\mathbf{V}(A) - \mathbf{V}(B)$. The vector running from the point A to the point B is usually denoted $\overrightarrow{AB}$. Note that $\overrightarrow{AB} = -\overrightarrow{BA}$.

Example: Find a vector running from the point $(-4, 0, 2)$ to the point

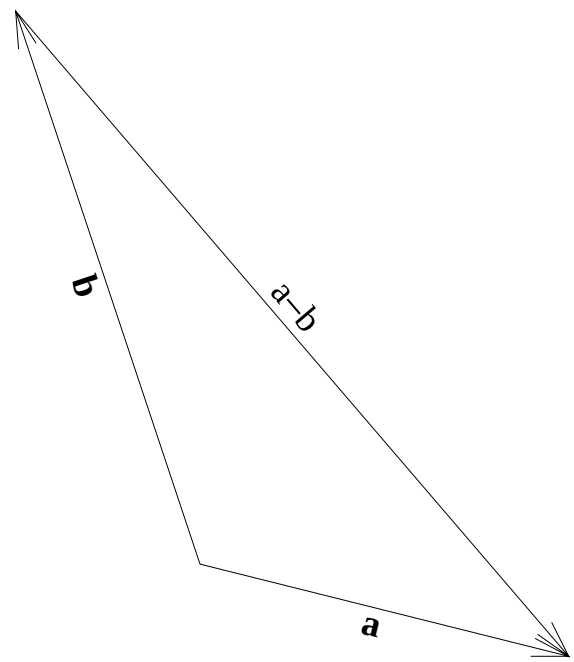

Figure 2.6: Subtracting **b** from **a**.

$(2, 1, 6)$. *Solution:* It may help to think of it this way: "From -4 to 2 is $2 - (-4) = 6$. From 0 to 1 is $1 - 0 = 1$. From 2 to 6 is $6 - 2 = 4$." The answer is therefore $\langle 6, 1, 4\rangle$.

Your turn: Find a vector running from the point $(5, -3 - 3)$ to the point $(0, 1, 10)$. Do it now. *Answer:* $\langle -5, 4, 13\rangle$.

Whenever three vectors form a triangle, each side is always the sum or difference of the other two.

2.3. Scalar multiplication.

Multiplying a vector by a scalar leaves the direction of the vector unchanged and increases or decreases the length. Multiplying by a number less than 1 makes the vector shorter; multiplying by a number greater than 1 makes the vector longer (Figure 2.7).

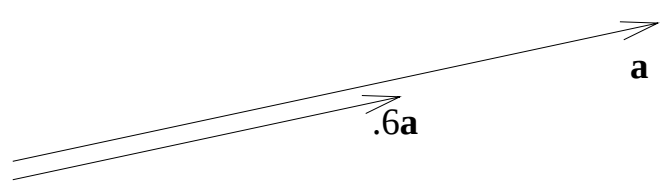

Figure 2.7: Multiplying a vector by 0.6.

Multiplying by a negative scalar reverses the direction of the vector (Figure 2.8).

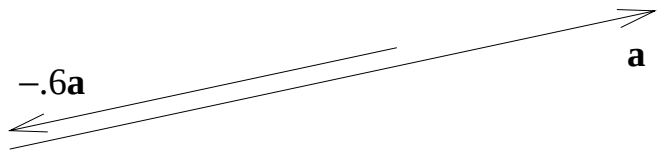

Figure 2.8: Multiplying a vector by -0.6.

One use for scalar multiplication is determining whether two vectors are parallel. They are parallel if and only if one is a scalar multiple of the other.

Example: Are the vectors $\langle 4, -1, 10\rangle$ and $\langle -20, 5, -50\rangle$ parallel? *Solution:* Since $-5\langle 4, -1, 10\rangle = \langle -20, 5, -50\rangle$, we see that the vectors are parallel, although in opposite directions.

Example: Are the vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ parallel? *Solution:* We must determine whether there exists a scalar s such that $s\langle 3, -1, 2\rangle = \langle 7, 1, 5\rangle$. By looking at the first component of the vectors, we see it is required that $(s)(3) = 7$; thus, if such an s exists, it must equal $7/3$. But then, looking at the second component of the vectors, we see $(s)(-1) = (7/3)(-1) \neq 1$. So no such scalar exists; and the vectors are not parallel.

Your turn: Are the vectors $\langle 4, 1, -2\rangle$ and $\langle 12, 3, -6\rangle$ parallel? *Answer:* Yes.

2.4. Magnitude of a vector.

In geometric terms, the magnitude of a vector is simply the length of the arrow.

A special term for a vector with magnitude 1 is a *unit vector*. One use of the magnitude is this: given a vector $\mathbf{a}$, find a unit vector $\mathbf{u}$ in the same direction. The formula for this is:

$$\mathbf{u} = \frac{1}{||\mathbf{a}||}\ \mathbf{a}.$$

We can prove this as follows. Since $\mathbf{u}$ is in the same direction as $\mathbf{a}$ there must be a scalar $s > 0$ such that $\mathbf{u} = s\mathbf{a}$. Using the algebraic properties of the magnitude from the last section, and requiring $\mathbf{u}$ to be a unit vector, we find:

$$1 = ||\mathbf{u}|| = ||s\mathbf{a}|| = |s|\ ||\mathbf{a}||;$$

Thus we must have $|s| = 1/||\mathbf{a}||$. We know $s > 0$, so $s = 1/||a||$.

Example: Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{a} = \langle 2, -6, 3\rangle$.
Solution: We calculate

$$\mathbf{u} = \frac{1}{||\mathbf{a}||}\,\mathbf{a} = \frac{1}{\sqrt{2^2 + (-6)^2 + 3^2}}\,\langle 2, -6, 3\rangle = \langle 2/7, -6/7, 3/7\rangle.$$

Your turn: Find a unit vector in the same direction as $\langle 5, 2, -1\rangle$. Do it now. *Answer:* $\langle 5/\sqrt{30}, 2/\sqrt{30}, -1/\sqrt{30}\rangle$.

Another common use of the magnitude is to find the distance between two points $\mathbf{a}$ and $\mathbf{b}$. Recall that the vector running from $\mathbf{a}$ to $\mathbf{b}$ is $\mathbf{b} - \mathbf{a}$. We can therefore calculate the distance as $||\mathbf{b} - \mathbf{a}||$.

Example: Find the distance between the points $(2, 0, 4)$ and $(10, 2, 2)$.
Solution: The answer is given by:

$$||\langle 10, 2, 2\rangle - \langle 2, 0, 4\rangle|| = ||\langle 8, 2, -2\rangle|| = \sqrt{8^2 + 2^2 + (-2)^2} = \sqrt{72} = 6\sqrt{2}.$$

Your turn: Find the distance between the points $(0, 4, 5)$ and $(1, 2, 7)$. Do it now. *Answer:* 3.

2.5. Dot product.

The geometric interpretation of the dot product $\mathbf{a} \cdot \mathbf{b}$ is this: Slide $\mathbf{a}$ and $\mathbf{b}$ so that they are joined at the tail. Let θ be the angle formed by $\mathbf{a}$ and $\mathbf{b}$ (Figure 2.9). Then

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}||\ \cos\theta.$$

In a sense, the dot product measures the "cooperation" between two vectors: a positive dot product indicates that the two vectors are roughly in the same direction; a negative dot product indicates that the two vectors are roughly in opposite directions.

There are several common uses of the dot product. One is to find the angle between two vectors. We simply solve for θ in the formula above:

$$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||\ ||\mathbf{b}||}.$$

Example: Find the angle between the vectors $2\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
Solution: We calculate

$$\begin{aligned}\theta &= \cos^{-1} \frac{(2\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})}{||2\mathbf{i} + \mathbf{j}||\ ||\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}||} \\ &= \cos^{-1} \frac{6}{\sqrt{5}\sqrt{21}} = 54.159^\circ.\end{aligned}$$

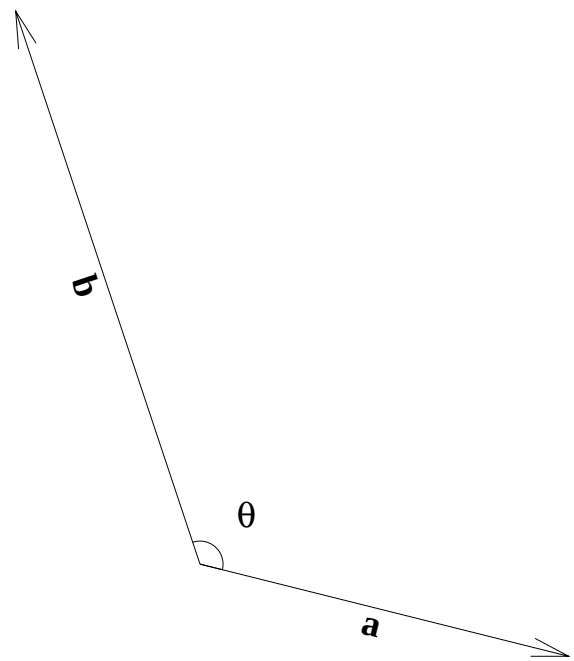

Figure 2.9: Two vectors and the angle between them.

Your turn: Find the angle between the vectors $\langle 4, 0, 5\rangle$ and $\langle -1, 7, -2\rangle$. Do it now. *Answer:* $107.3°$.

A special case of finding the angle is verifying whether or not two vectors are perpendicular. Since $\cos 90° = 0$, two vectors are perpendicular if and only if their dot product is zero.

Example: Are the vectors $2\mathbf{i}+3\mathbf{j}$ and $\mathbf{i}-3\mathbf{j}-\mathbf{k}$ perpendicular? *Solution:* Since

$$(2\mathbf{i} + 3\mathbf{j}) \cdot (\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = (2)(1) + (3)(-3) + (0)(-1) = -7 \neq 0,$$

the answer is NO.

Your turn: Are the vectors $\langle 1, -3, 7\rangle$ and $\langle -1, 2, 1\rangle$ perpendicular? *Answer:* Yes.

Another common use for the dot product is to calculate the projection of one vector onto another. There are two flavors of this calculation, depending on whether we want our answer in scalar or vector form. We consider the scalar version first.

The scalar projection p of a vector $\mathbf{a}$ onto another vector $\mathbf{b}$ is the length of the "shadow" cast by $\mathbf{a}$ onto $\mathbf{b}$ by a beam of light perpendicular to $\mathbf{b}$ (Figure 2.10). We can obtain a formula for the distance p using trigonometry:

$$p = ||\mathbf{a}|| \cos\theta.$$

Note how similar this is to our previous formula $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos\theta$. We obtain immediately

$$p = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||}.$$

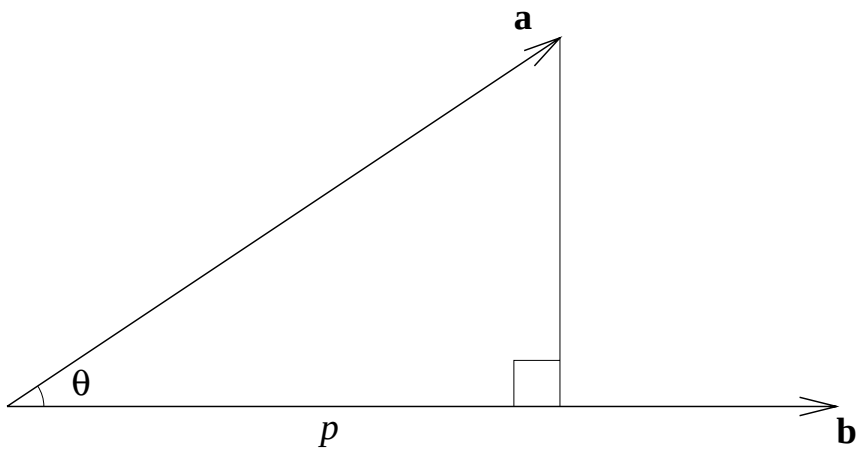

Figure 2.10: Projection of **a** onto **b**.

Example: Find the scalar projection of $\langle -3, -1, 5\rangle$ onto $\langle 1, 1, 2\rangle$. *Solution:* The answer is

$$p = \frac{\langle -3, -1, 5\rangle \cdot \langle 1, 1, 2\rangle}{||\langle 1, 1, 2\rangle||} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

Your turn: Find the scalar projection of $\langle 2, -1, 1\rangle$ onto $\langle 1, 0, 3\rangle$. Do it now. *Answer:* $5/\sqrt{10}$.

The vector projection $\mathbf{p}$ of $\mathbf{a}$ onto $\mathbf{b}$ is a vector with length p, in the direction of $\mathbf{b}$. We can obtain $\mathbf{p}$ as the product of p and a unit vector $\mathbf{u}$ in the direction of $\mathbf{b}$, each of which we know how to obtain separately:

$$\begin{aligned}\mathbf{p} = p\mathbf{u} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||}\right)\left(\frac{1}{||\mathbf{b}||}\ \mathbf{b}\right) \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||^2}\mathbf{b}.\end{aligned}$$

Example: Find the vector projection of $\langle -3, -1, 5\rangle$ onto $\langle 1, 1, 2\rangle$. *Solution:* The answer is

$$\mathbf{p} = \frac{\langle -3, -1, 5\rangle \cdot \langle 1, 1, 2\rangle}{||\langle 1, 1, 2\rangle||^2}\langle 1, 1, 2\rangle = \frac{6}{6}\langle 1, 1, 2\rangle = \langle 1, 1, 2\rangle.$$

You can verify that $||\mathbf{p}|| = p$.

Your turn: Find the vector projection of $\langle 2, -1, 1\rangle$ onto $\langle 1, 0, 3\rangle$. Do it now. *Answer:* $\langle 1/2, 0, 3/2\rangle$.

2.6. Cross product.

The geometric interpretation of the cross product $\mathbf{a} \times \mathbf{b}$ is this: Slide $\mathbf{a}$ and $\mathbf{b}$ so that they are joined at the tail. We can define a parallelogram with two edges determined by $\mathbf{a}$ and $\mathbf{b}$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is a vector $\mathbf{c}$ with the following properties:

(1) The magnitude of $\mathbf{c}$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

(2) The direction of $\mathbf{c}$ is perpendicular to the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. One could also say that $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Actually, these two conditions do not quite pin down $\mathbf{c}$, because $-\mathbf{c}$ satisfies the same conditions as $\mathbf{c}$. In order to choose which of these two opposite vectors is the cross product, we rely on the *right-hand rule*: if you curl the fingers of your right hand (make sure to use the correct hand) from $\mathbf{a}$ into $\mathbf{b}$, your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$ (Figure 2.11). In Figure 2.1, because $\mathbf{a} \times \mathbf{b}$ points upward, $\mathbf{a}$ must lie in front of $\mathbf{b}$.

The combination of the area and the perpendicular direction in one vector actually proves to be a great convenience, because areas in three dimensions have various orientations, just as vectors do. This is a special case of a general principle, which is worth emphasizing:

Usually, the best way to specify the orientation of a surface in space is by the perpendicular (also called normal*) vector.*

We shall see this principle in action when we consider more elaborate geometrical problems.

One obvious use of the cross product is to find areas.

Example: Find the area of the parallelogram defined by $\langle 2, 0, 2\rangle$ and $\langle -1, 5, 1\rangle$. *Solution:* The answer is

$$A = ||\langle 2, 0, 2\rangle \times \langle -1, 5, 1\rangle|| = ||\langle -10, -4, 10\rangle|| = \sqrt{216} = 6\sqrt{6}.$$

Your turn: Find the area of the parallelogram defined by $\langle 1, 4, -2\rangle$ and $\langle 3, 1, 0\rangle$. Do it now. *Answer:* $\sqrt{161}$.

Another easy use is to find a vector perpendicular to two given vectors.

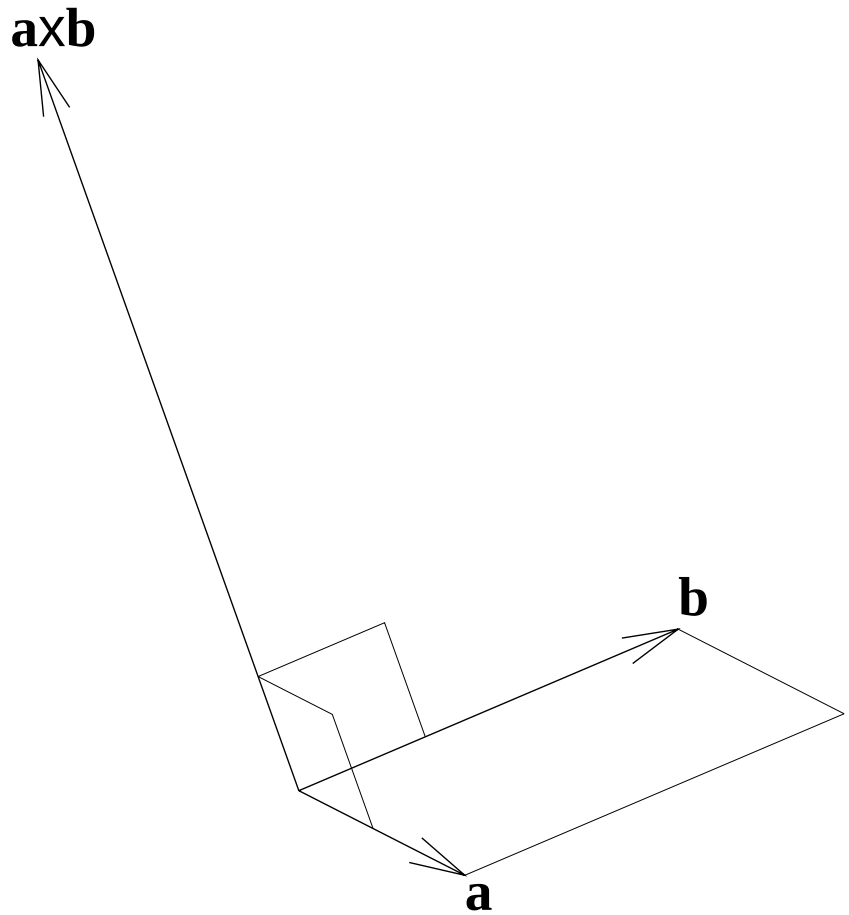

Figure 2.11: The cross product of **a** and **b**.

Example: Find a vector perpendicular to both $\langle 2, 0, 2\rangle$ and $\langle -1, 5, 1\rangle$. *Solution:* The answer is

$$\langle 2, 0, 2\rangle \times \langle -1, 5, 1\rangle = \langle -10, -4, 10\rangle.$$

Your turn: Find a vector perpendicular to both $\langle 3, 5, -2\rangle$ and $\langle -1, -1, 2\rangle$. Do it now. *Answer:* $\langle 8, -4, 2\rangle$.

Another use for the cross product, less obvious, is to determine whether two vectors are parallel. We have already seen how to use the scalar product to answer this question; and usually the scalar product is easier to use, but in some situations, the cross product is a better tool. Two vectors are parallel if and only if their cross product is zero. This makes sense if you think about what happens to the parallelogram defined by the two vectors as they become parallel.

Example: Are the vectors $\langle 4, -1, 10\rangle$ and $\langle -20, 5, -50\rangle$ parallel? *Solu-*

tion: Since

$$\langle 4, -1, 10\rangle \times \langle -20, 5, -50\rangle = \langle 0, 0, 0\rangle,$$

the answer is YES.

Your turn: Are the vectors $\langle 2, -1, 0\rangle$ and $\langle -10, 5, 0\rangle$ parallel? *Answer:* Yes.

Two vectors determine a parallelogram in space. In the same way, three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ determine a *parallelepiped* (the three-dimensional equivalent of a parallelogram) in space. (Figure 2.12) The cross product and dot product can be used together to find the volume of this parallelepiped: the volume is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$. This combination is sometimes called the *triple scalar product.*

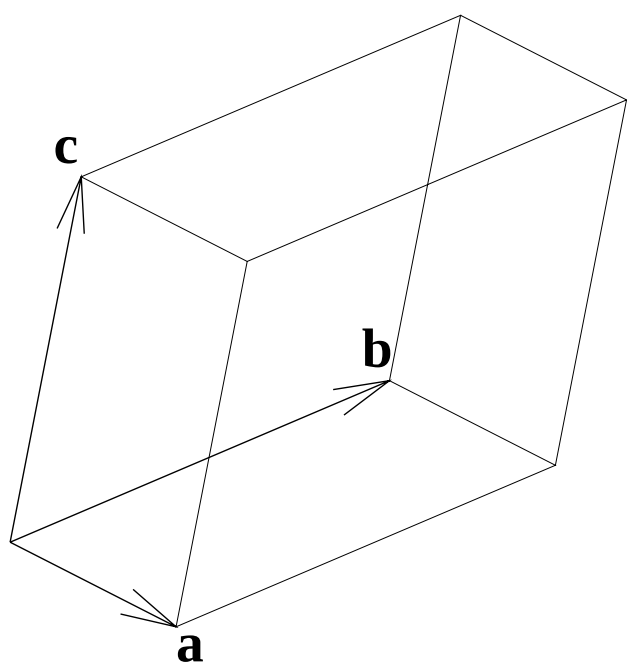

Figure 2.12: The parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

The volume of the parallelepiped is of course independent of the order of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; and the value of the triple scalar product is likewise independent, except for the sign. We have:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} =$$

$$-(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}.$$

The orderings of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ which yield a positive triple scalar product are called "right-handed"; and those which yield a negative triple scalar product are called "left-handed."

The volume of the parallelepiped is zero if and only if the three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ lie in the same plane. Thus three vectors are coplanar if and only if their triple scalar product is zero.

2.7. Summary.

Learn the following by heart:

- How to convert two-dimensional vectors back and forth between Cartesian and polar form.
- Be able to explain in geometric terms the effects of the following operations: vector addition, vector subtraction, scalar multiplication, magnitude, dot product, cross product.
- Vectors are $\mathbf{a}$ and $\mathbf{b}$ are parallel if there exists a scalar s such that $\mathbf{a} = s\mathbf{b}$ or if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- To find a vector $\mathbf{b}$ parallel to a given vector: let $\mathbf{b} = s\mathbf{a}$; choose s so as to satisfy any other requirements for $\mathbf{b}$.
- Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$.
- The angle θ between vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||\ ||\mathbf{b}||}.$$

- The scalar projection p of vector $\mathbf{a}$ onto vector $\mathbf{b}$ is given by

$$p = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||};$$

  be able to describe geometrically what this means.
- The vector projection $\mathbf{p}$ of vector $\mathbf{a}$ onto vector $\mathbf{b}$ is given by

$$\mathbf{p} = \frac{\langle -3, -1, 5\rangle \cdot \langle 1, 1, 2\rangle}{||\langle 1, 1, 2\rangle||^2}\langle 1, 1, 2\rangle = \frac{6}{6}\langle 1, 1, 2\rangle = \langle 1, 1, 2\rangle;$$

  be able to describe geometrically what this means.
- A unit vector is a vector with magnitude 1.
- A vector $\mathbf{b}$ in the same direction as vector $\mathbf{a}$ with given magnitude s is given by $s\mathbf{a}/||\mathbf{a}||$.

- A vector $\mathbf{a}$ running from point (x_1, y_1, z_1) to point (x_2, y_2, z_2) is given by $\langle x_2 - x_1, y_2 - y_1, z_2, z_1 \rangle$.
- The area of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $||\mathbf{a} \times \mathbf{b}||$.
- A vector perpendicular to vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $\mathbf{a} \times \mathbf{b}$.
- The volume of the parallelepiped formed by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.
- Three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{0}$.

2.8. Exercises.

1. What does the relationship $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$ (from the last section) mean in geometric terms?

2.–7. Tell whether the given pairs of vectors are parallel or not.

2. $\langle -3, 4, 0\rangle$ and $\langle 6, -8, 2\rangle$
3. $\langle 1, 1, 2\rangle$ and $\langle 10, 10, 20\rangle$
4. $\langle 4, 2, 3\rangle$ and $\langle 24, 12, 18\rangle$
5. $\langle 5, 0, 2\rangle$ and $\langle 6, 0, -15\rangle$
6. $\langle 2, 1, 9\rangle$ and $\langle -3, 3, 3\rangle$
7. $\langle 1, 0, 0\rangle$ and $\langle 5, 1, 2\rangle$

8.–13. For each pair of vectors in Problems 2.–7., determine whether they are perpendicular.

14.–19. Find the angle between each pair of vectors in Problems 2.–7.

20.–25. For each pair of vectors in Problems 2.–7., find the scalar projection of the first vector onto the second.

26.–31. For each pair of vectors in Problems 2.–7., find the vector projection of the first vector onto the second.

32.–37. Find a unit vector in the same direction as each of the following vectors:

32. $\langle 3, 4, 12\rangle$
33. $\langle -1, 3, 3\rangle$
34. $\langle 10, 2, -1\rangle$
35. $\langle -8, 2, 7\rangle$
36. $\langle 0, 3, 0\rangle$
37. $\langle 8, 8, 2\rangle$

38.–43. For each pair of points, find a vector running from the first point to the second.

38. $(-3, 4, 0)$ to $(6, -8, 2)$
39. $(1, 1, 2)$ to $(10, 10, 20)$
40. $(4, 2, 3)$ to $(24, 12, 18)$
41. $(5, 0, 2)$ to $(6, 0, -15)$
42. $(2, 1, 9)$ to $(-3, 3, 3)$
43. $(1, 0, 0)$ to $(5, 1, 2)$

44. Explain why $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ in geometric terms.

45. Explain why $(\mathbf{i} \times \mathbf{j}) \cdot \mathbf{k} = 1$ in geometric terms.

46. How can you use one of the other vector operations to check the result of a cross-product computation?

47.–52. Find the area of the parallelogram determined by each pair of vectors in Problems 2.–7.

53.–58. Find a vector perpendicular to each pair of vectors in Problems 2.–7.

59.–64. Find the volume of the parallelepiped determined by each triple of vectors.

59. $\langle -3, 4, 0\rangle$, $\langle 6, -1, 2\rangle$, and $\langle 1, 0, 2\rangle$
60. $\langle 1, 1, 2\rangle$, $\langle 0, 10, 20\rangle$, and $\langle -1, 9, 18\rangle$
61. $\langle 4, 2, 3\rangle$, $\langle 24, 12, 18\rangle$, and $\langle 3, 5, 2\rangle$
62. $\langle 5, 0, 2\rangle$, $\langle -2, 1, 1\rangle$ and $\langle 6, 0, -15\rangle$

63. $\langle 2, 1, 9\rangle$, $\langle -1, 4, 12\rangle$, and $\langle -3, 3, 3\rangle$
64. $\langle 1, 0, 0\rangle$, $\langle 5, 1, 2\rangle$, and $\langle 0, 2, 0\rangle$

65.–70. Determine whether each triple of vectors in Problems 59.–64. is coplanar.

71.–76. Find a two-dimensional vector with the given length r and making the given angle θ with the positive x-axis.

71. $r = 4$, $\theta = 27°$
72. $r = 7$, $\theta = 5\pi/4$
73. $r = 1$, $\theta = -\pi/3$
74. $r = 5$, $\theta = 111°$
75. $r = 10$, $\theta = 90°$
76. $r = 0$, $\theta = 40°$

77. An airplane pilot wishes to maintain a true course in the direction 240° with a ground speed of 400 mi/hr when the wind is blowing directly north at 50 mi/hr. Find the required airspeed and compass heading.

78. Under what conditions is $||\mathbf{a} + \mathbf{b}|| = ||\mathbf{a}|| + ||\mathbf{b}||$?

79. Find the angle between the vector running from the point $(1, 1, 1)$ to the point $(-1, 1, -1)$ and that running from the point $(1, 1, 1)$ to the point $(1, -1, -1)$.

80. *Coulomb's Law* states that the magnitude of the force of attraction between two oppositely charged particles is directly proportional to the product of the magnitudes q_1 and q_2 of the charges and inversely proportional to the square of the distance d between them. Show that if a point with charge $+q$ is fixed at point A and a particle of charge -1 is placed at point B, then the force of attraction $\mathbf{F}$ at A is given by

$$\mathbf{F} = \frac{kq}{||\overrightarrow{BA}||^3}\overrightarrow{BA}$$

for some negative constant k.

81. If AB is a diameter of a sphere with center O and radius r and if P is a third point on the sphere, use vectors to show that APB is a right triangle. (*Hint:* let $\mathbf{v}_1 = \overrightarrow{OA}$, $\mathbf{v}_2 = \overrightarrow{OP}$, and write $\overrightarrow{PA}$ and $\overrightarrow{PB}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$.)

2.9. Answers.

1. The length of one side of a triangle is no more than the sum of the lengths of the other two sides.

2. No.	3. Yes.	4. Yes.
5. No.	6. No.	7. No.
8. No.	9. No.	10. No.
11. Yes.	12. No.	13. No.
14. $168.69°$	15. $0°$	16. $0°$
17. $90°$	18. $60.13°$	19. $24.09°$
20. $-50/\sqrt{104}$	21 $\sqrt{6}$	22. $\sqrt{29}$
23. 0	24. $24/\sqrt{27}$	25. $5/\sqrt{30}$

26. $\langle -75/26, 50/13, -25/26\rangle$	27. $\langle 1, 1, 2\rangle$
28. $\langle 4, 2, 3\rangle$	29. $\langle 0, 0, 0\rangle$
30. $\langle -8/3, 8/3, 8/3\rangle$	31. $\langle 5/6, 1/6, 1/3\rangle$
32. $\langle 3/13, 4/13, 12/13\rangle$	33. $\langle -1/\sqrt{19}, 3/\sqrt{19}, 3/\sqrt{19}\rangle$
34. $\langle 10/\sqrt{105}, 2/\sqrt{105}, -1/\sqrt{105}\rangle$	35. $\langle -8/\sqrt{117}, 2/\sqrt{117}, 7/\sqrt{117}\rangle$
36. $\langle 0, 1, 0\rangle$	37. $\langle 4/\sqrt{33}, 4/\sqrt{33}, 1/\sqrt{33}\rangle$

38. $\langle 9, -12, 2\rangle$	39. $\langle 9, 9, 18\rangle$	40. $\langle 20, 10, 15\rangle$
41. $\langle 1, 0, -17\rangle$	42. $\langle -5, 2, -6\rangle$	43. $\langle 4, 1, 2\rangle$

44. The parallelogram formed by $\mathbf{i}$ and $\mathbf{j}$ is a 1×1 square. The area of this square is 1, which is the magnitude of $\mathbf{i} \times \mathbf{j}$, and the direction is perpendicular to the (x, y)-plane; i.e., along the z-axis.

45. The parallelepiped formed by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is a $1 \times 1 \times 1$ cube. The volume of this cube is 1.

46. Since $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, the result $\mathbf{c}$ should satisfy $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = 0$. Otherwise, the result must be incorrect (although even if both dot products are zero, the result could be incorrect by a scalar multiple).

47. 10	48. 0	49. 0
50. 87	51. $3\sqrt{194}$	52. $\sqrt{5}$

Problems 53-58 have many other possible answers.

53. $\langle 8, 6, 0\rangle$	54. $\langle 1, -1, 0\rangle$	55. $\langle 1, -2, 0\rangle$
56. $\langle 0, 1, 0\rangle$	57. $\langle -24, -33, 9\rangle$	58. $\langle 0, -2, 1\rangle$

59. 34	60. 0	61. 0
62. 87	63. 0	64. 4

65. No.	66. Yes.	67. Yes.
68. No.	69. No.	70. No.
71. $3.56\mathbf{i} + 1.82\mathbf{j}$	72. $-7/\sqrt{2}\mathbf{i} - t/\sqrt{2}\mathbf{j}$	73. $1/2\mathbf{i} - \sqrt{3}/2\mathbf{j}$
74. $-1.79\mathbf{i} + 4.67\mathbf{j}$	75. $10\mathbf{j}$	76. $\mathbf{0}$

77. airspeed 427.2 mi/hr, compass heading 234.2°
78. when **a** and **b** are in the same direction
79. 60°

3. Vectors and geometry.

In Section 1, you acquired the basic tools for manipulating vectors. In Section 2, you learned the basic functions of these tools. In this section you will combine the tools in many different ways to solve a wide variety of geometric problems.

You will most likely find this section quite a bit more demanding than the two preceding. Up to now, most of your exercises have been closely patterned after examples in the text. For example, the exercises in Section 2 are the vector equivalent of "hammer this nail", "drive this screw", "cut this board in half." The exercises in this section are more like "build a birdhouse", or maybe even "build a wooden replica of Saint Peter's basilica."

It is impossible to give examples of every type of problem that you may encounter, and you should focus on understanding the sequence of steps used in each example, and being able to make variations on the pattern, rather than memorizing the solution to any particular problem.

Your exercises will mostly be types of problems unlike any examples in the text. You may need to spend several minutes of hard thought (at least) on any of them before knowing how to proceed. Working together with one or two classmates may be helpful.

3.1. Planes.

A particular geometric problem which arises fairly frequently, either on its own or as part of another problem, is determining the equation of a plane in three dimensions. It is a good idea to memorize the method of solving this particular problem.

A standard definition for us is to let $\mathbf{r}$ represent the position vector $\langle x, y, z\rangle$ or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ of the point (x, y, z); thus any statement about x, y, and z (such as the equation of a surface) can be recast as a statement about $\mathbf{r}$ and vice versa. Usually we shall be translating statements about $\mathbf{r}$ into terms of x, y, and z.

Example: Put the statement

$$(4\mathbf{i} - \mathbf{k}) \cdot \mathbf{r} = ||\mathbf{r}||$$

in terms of x, y, and z. *Solution:* Using our formulas for vector operations, we have:

$$(4\mathbf{i} - \mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = ||x\mathbf{i} + y\mathbf{j} + z\mathbf{k}||,$$

$$4x - z = \sqrt{x^2 + y^2 + z^2}.$$

Your turn: Put the statement $\mathbf{r} \cdot (\mathbf{r} - \mathbf{j}) = 7$ in terms of x, y, and z. Do it now. *Answer:* $x^2 + y^2 - y + z^2 = 7$.

In general, the easiest way to find the equation for a plane is to start with two things:

(1) : A vector $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ perpendicular to the plane (the *normal* vector), and

(2) : A point $P = (p_1, p_2, p_3)$ on the plane.

Our problem now, is to find a condition on (x, y, z) which guarantees that it will also lie on the plane. A little geometric thinking shows that (x, y, z) lies on the plane if and only if the vector from P to (x, y, z) is perpendicular to $\mathbf{n}$ (Figure 3.1). This breaks the problem down into two steps, each of which we have already learned:

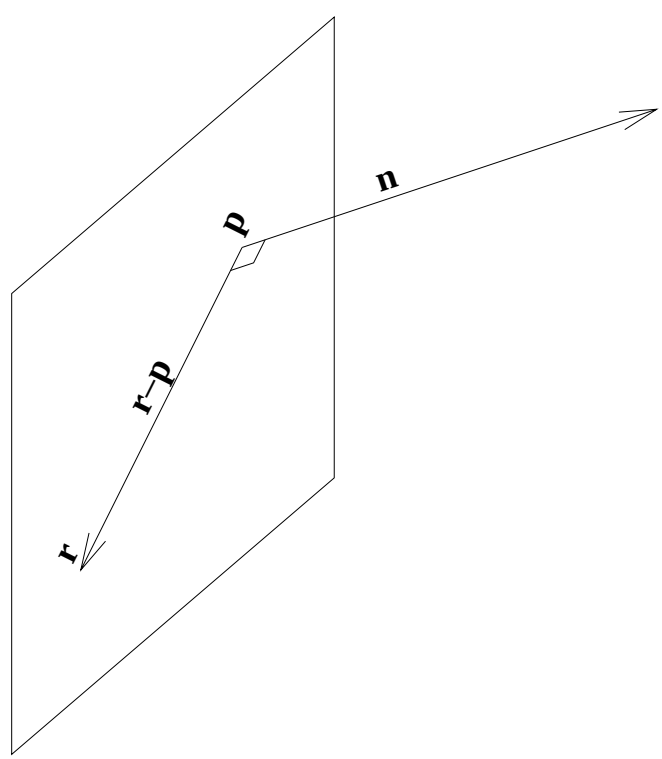

Figure 3.1: A plane with normal vector and a vector in the plane.

Step 1: Find a vector from P to (x, y, z). This is done by subtracting $\langle x, y, z \rangle - \mathbf{V}(P) = \mathbf{r} - \mathbf{p}$, where $\mathbf{p} = \langle p_1, p_2, p_3 \rangle$.

Step 2: Require that the vector from Step 1 be perpendicular to $\mathbf{n}$. This is done by requiring $\mathbf{n} \cdot (\mathbf{r} - \mathbf{p}) = 0$.

Thus, in vector form, the equation for a plane is:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{p}) = 0,$$

or

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{p}.$$

Let's put this in terms of x, y, and z. We'll do a little more manipulation to put the equation in a standard form.

$$\langle n_1, n_2, n_3 \rangle \cdot \langle x, y, z \rangle = \langle n_1, n_2, n_3 \rangle \cdot \langle p_1, p_2, p_3 \rangle,$$

$$n_1 x + n_2 y + n_3 z = n_1 p_1 + n_2 p_2 + n_3 p_3.$$

The above is our "standard" form for the equation of a plane:

$$Ax + By + Cz = D.$$

All planes have equations which can be put in this form, and any equation of this form describes a plane in three dimensions. It's like the three-dimensional equivalent of the equation of a line.

Example: Find an equation for the plane perpendicular to the vector $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and containing the point $\langle 1, -1, 7 \rangle$. *Solution:* The equation is:

$$\langle 5, -3, 2 \rangle \cdot \langle x, y, z \rangle = \langle 5, -3, 2 \rangle \cdot \langle 1, -1, 7 \rangle,$$

$$5x - 3y + 2z = (5)(1) + (-3)(-1) + (2)(7) = 22.$$

3.2. The inverse problem.

Notice that it is easy to recover $\mathbf{n}$ from the final equation: just take the coefficients of x, y, and z.

Example: Find a normal vector to the plane $8x - 3y - 2z = 10$. *Solution:* $\mathbf{n} = \langle 8, -3, -2 \rangle$.

Finding a point on a plane is also easy. There are infinitely many possible solutions.

Example: Find a point on the plane $8x - 3y - 2z = 10$. *Solution:* We can let $x = 0$ and $y = 0$ and solve for $z = -5$ to get the point $\langle 0, 0, -5 \rangle$.

3.2. Working with planes.

When questions arise about the orientation of the plane, the angle between the plane and something else, etc., it is usually easiest to work with the normal vector to the plane.

Example: Find the angle between the plane $8x - 3y - 2z = 10$ and the vector $\mathbf{a} = 6\mathbf{i} - 3\mathbf{j}$. *Solution:* (You get to fill in the steps now.) First find the normal vector $\mathbf{n}$ to the plane:

$$\mathbf{n} =$$

The angle ϕ between the plane and $\mathbf{a}$ is the complement of the angle θ between $\mathbf{n}$ and $\mathbf{a}$. So find the angle θ between $\mathbf{n}$ and $\mathbf{a}$:

$$\theta =$$

Now find ϕ:

$$\phi =$$

(The correct answer is $\phi = 75.5°$.

Often the two things we need to determine a plane (the normal vector and the point) are not given to us directly, so we need to do some figuring before getting the equation of the plane.

Example: Find an equation of the plane containing the points $(0, 2, 2)$, $(3, -1, 0)$, and $(1, 4, 4)$.

Solution: (Again, you to fill in the steps.) To find the equation, we need a normal vector $\mathbf{n}$ to the plane and a point $\mathbf{p}$ on the plane. Of points we have plenty—but the normal vector $\mathbf{n}$ is not given. How to find it?

Consider that $\mathbf{n}$ must be perpendicular to any vector lying in the plane. The vector running from any of our given points to any other lies in the plane. Find a vector $\mathbf{a}$ running from one of our given points to another:

$$\mathbf{a} =$$

Find another vector $\mathbf{b}$ running between another pair of given points:

$$\mathbf{b} =$$

Any two pairs of points—as long as all three points are used—will work fine.

We now require that $\mathbf{n}$ be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Remember this can be done by the cross product. Calculate $\mathbf{n}$ as $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{n} =$$

And find the equation of the plane:

(The correct answer is $-2x - 8y + 9z = 2$ or something equivalent.)

Another solution: This problem, like most others, can be solved by more than one approach. As an illustration, let's go through a solution by a different method.

This time, we start by observing that $\mathbf{r}$ is on the plane determined by the points $(0, 2, 2)$, $(3, -1, 0)$, and $(1, 4, 4)$ provided that all four points are coplanar. We have not learned how to determine whether four points are coplanar, but we have learned that three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are coplanar provided that the triple scalar product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is zero.

We will start by determining three vectors in the plane by subtracting the coordinates of our four points $(0, 2, 2)$, $(3, -1, 0)$, $(1, 4, 4)$, and $\mathbf{r}$. There are many ways to combine the four points in pairs. Just make sure not to use the same pair twice; use each point at least once; and also, the computations will be much easier if you use the point $\mathbf{r}$ only once.

Find a vector $\mathbf{a}$ running from one of the given points to another:

$$\mathbf{a} =$$

Find another vector $\mathbf{b}$ running between a different pair of points:

$$\mathbf{b} =$$

And find a third vector $\mathbf{c}$ running between a third pair of points:

$$\mathbf{c} =$$

Calculate $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} =$$

Now calculate $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} =$$

And set the result to zero to obtain the equation of the plane:

3.3. Another geometric example.

As another example of combining vector tools to solve geometric problems, we consider the problem of finding the distance from the point $(2, -1, 1)$ to the line passing through the points $(0, 2, -4)$ and $(1, 3, 3)$.

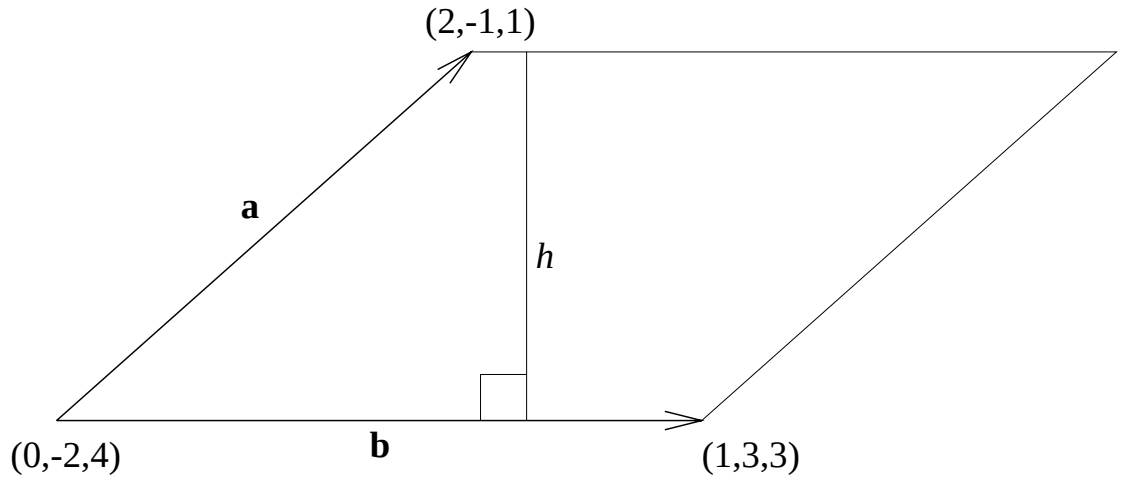

Figure 3.2: Finding distance from a point to a line.

Solution: We could easily find the distance from $(2, -1, 1)$ to $(0, 2, -4)$ or from $(2, -1, 1)$ to $(1, 3, 3)$, but these are not what we're looking for. What we need is the *perpendicular* distance from the point to the line. For this problem, an indirect approach works well.

Consider the parallelogram with base running from $(0, 2, -4)$ to $(1, 3, 3)$ and another side running from $(0, 2, -4)$ to $(2, -1, 1)$. Let $\mathbf{b}$ be the vector running along the base and $\mathbf{a}$ be the vector running up the side. We have learned how to calculate the area of the parallelogram as $||\mathbf{a} \times \mathbf{b}||$. With high-school geometry, we can also calculate the area as length of base times height. But the height h is exactly the distance we want to find. So our strategy is to calculate the area of the parallelogram by the cross product, find the length of the base, and divide to find the distance h.

Find the vector $\mathbf{b}$ running from $(0, 2, -4)$ to $(1, 3, 3)$:

$$\mathbf{b} =$$

Find the vector $\mathbf{a}$ running from $(0, 2, -4)$ to $(2, -1, 1)$:

$$\mathbf{a} =$$

Find the area $||\mathbf{a} \times \mathbf{b}||$ of the parallelogram:

$$||\mathbf{a} \times \mathbf{b}|| =$$

Find the length $||\mathbf{b}||$ of the base:

$$||\mathbf{b}|| =$$

Divide to find the distance h:

$$h =$$

3.4. Problems.

Some of the following problems can be done by imitating examples in this section. Others require you to synthesize the basic facts about planes with other things you have learned about vectors. Some of the problems don't involve planes at all, but can be solved with the vector tools you have studied.

In Problems 1-6, find an equation for the plane with the given normal vector $\mathbf{n}$ and containing the given point P.

1. $\mathbf{n} = \langle -3, 4, 0\rangle$, $P = (6, -8, 2)$
2. $\mathbf{n} = \langle 1, 1, 2\rangle$, $P = (10, 10, 20)$
3. $\mathbf{n} = \langle 4, 2, 3\rangle$, $P = (24, 12, 18)$
4. $\mathbf{n} = \langle 5, 0, 2\rangle$, $P = (6, 0, -15)$
5. $\mathbf{n} = \langle 2, 1, 9\rangle$, $P = (-3, 3, 3)$
6. $\mathbf{n} = \langle 1, 0, 0\rangle$, $P = (5, 1, 2)$

7. Are the planes $4x - 2y - 2z = 10$ and $-2x + y + z = 6$ parallel?

8. Are the planes $2x + y + z = 9$ and $2x + 3y + 3z = 7$ parallel?

9. Are the planes $2x - y - 2z = 0$ and $10x + 10z = 1$ perpendicular?

10. Are the planes $4x - 4y - 4z = 0$ and $2x + y + 8z = 1$ perpendicular?

11. Find the angle between the planes $2x + y - z = 0$ and $4x - 4y + 7z = 3$.

12. Find the angle between the planes $6x + 2y = 4$ and $x + y + z = 2$.

13. Find the angle between the plane $x + y + z = 0$ and the x-axis.

14. Find the distance between the point $(2, 1, 1)$ and the plane $x + 5y - z = 1$.

15. Find the distance between the origin and the plane $5x + 3y + z = 9$.

16. Find an equation for the plane containing the points $(0, 1, 1)$, $(1, 0, 3)$, and $(5, 5, 2)$.

17. Find an equation for the plane containing the points $(-3, 1, 1)$, $(9, 1, 3)$, and $(7, 0, 0)$.

18. Find an equation for the sphere with center $(2, 8, 1)$ and radius 8.

19. Find an equation for the cone whose axis is the vector $\langle 4, 1, 1\rangle$, apex at the origin, and vertex angle $90°$.

20. Find the angle between the plane $x + y = 3$ and the line passing through the points $(0, 0, 1)$ and $(1, 1, 1)$.

3.5. Answers.

1. $-3x + 4y = -50$
2. $x + y + 2z = 60$
3. $4x + 2y + 3z = 174$
4. $5x + 2z = 0$
5. $2x + y + 9z = 24$
6. $x = 5$
7. Yes.
8. No.
9. Yes.
10. No.
11. $97.8°$
12. $43.1°$
13. $35.3°$
14. $5/(3\sqrt{3})$
15. $9/\sqrt{35}$
16. $-x + y + z = 2$
17. $x + 16y - 6z = 7$
18. $(x-2)^2 + (y-8)^2 + (z-1)^2 = 64.$
19. $2(4x + y + z)^2 = 18(x^2 + y^2 + z^2).$
20. $90°$

3.6. More Problems.

1. Find three distinct points in the plane with equation $2x-3y+z=1$.

2. Find three distinct points in the plane with equation $x+7y=12$.

3. Find two distinct vectors lying in the plane with equation $x-5y-z=2$.

4. Find two distinct vectors lying in the plane with equation $y+2z=0$.

5. Find the distance between the point $(0,2,1)$ and the line passing through the points $(0,0,0)$ and $(2,2,1)$.

6. Find the distance between the point $(10,1,1)$ and the line passing through the points $(3,-1,2)$ and $(0,1,0)$.

7. Find the distance between the line passing through the points $(0,0,0)$ and $(1,1,1)$ and the line passing through the points $(4,9,0)$ and $(7,0,0)$.

8. Find the distance between the line passing through the points $(2,2,0)$ and $(-2,0,1)$ and the line passing through the points $(-3,-3,-3)$ and $(1,2,2)$.

9. Find an equation for the plane containing the points $(2,2,0)$, $(-2,0,1)$ and $(-3,-3,-3)$.

10. Find an equation for the plane containing the points $(4,0,1)$, $(2,2,0)$ and $(0,4,-1)$.

11. Find an equation for the cylinder whose axis is the line passing through the points $(0,0,0)$ and $(4,0,1)$ and with radius 1.

12. Find an equation for the cylinder whose axis is the line passing through the points $(1,-1,1)$ and $(2,2,9)$ and with radius 10.

13. Find the angle between the line passing through the points $(0,0,0)$ and $(1,1,1)$ and the line passing through the points $(4,9,0)$ and $(7,0,0)$.

14. Find the angle between the line passing through the points $(2,2,0)$ and $(-2,0,1)$ and the line passing through the points $(-3,-3,-3)$ and $(1,2,2)$.

15. Are the points $(2,1,1)$, $(0,0,1)$, $(4,2,5)$, and $(6,3,5)$ coplanar?

16. Are the points $(3,-1,0)$, $(4,-1,4)$, $(2,2,2)$, and $(1,-3,0)$ coplanar?

3.7. More Answers.

1. $(.1/2, 0, 0)$, $(0, -1/3, 0)$, $(0, 0, 1)$; many other possible answers.
2. $(12, 0, 0)$, $(12, 0, 10)$, $(12, 0, 2)$; many other possible answers.
3. $\langle 2, 0, 2\rangle$, $\langle 7, 1, 2\rangle$; many other possible answers.
4. $\langle 1, 0, 0\rangle$, $\langle 2, 2, -1\rangle$; many other possible answers.
5. $\sqrt{20}/3$
6. $\sqrt{693}/\sqrt{17}$
7. $21/\sqrt{26}$
8. $3/\sqrt{105}$
9. $11x - 17y + 10z = -12$
10. This is a trick question; because the three points given are colinear, there are many different planes containing all three points. You probably encountered difficulty if you tried to solve this problem the same way as Problem 9.
11. $17y^2 + (4z - x)^2 = 17$
12. $(8y - 3z + 11)^2 + (z - 8x + 7)^2 + (3x - y - 4)^2 = 7400.$
13. 68.6° or 111.4°
14. 55.7° or 124.3°
15. Yes
16. No

PRACTICE TEST

Problems 1-15: Let $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \langle 2, 5, 1\rangle$, $\mathbf{c} = \langle -1, 1, 0\rangle$, and $\mathbf{r} = \langle x, y, z\rangle$. Find the following if possible, or state so, if impossible:

1. $\mathbf{a} + 3\mathbf{b} - \mathbf{c}$
2. $||\mathbf{r} + 2\mathbf{a}||$
3. $\mathbf{a} \cdot \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b})/\mathbf{c}$
5. $\mathbf{a} \times \mathbf{b}$
6. $(\mathbf{a} \cdot \mathbf{c}) \times \mathbf{b}$
7. $(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$
8. $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b}$
9. The angle between the vectors $\mathbf{a}$ and $\mathbf{c}$.
10. The scalar projection of $\mathbf{r}$ onto $\mathbf{a}$.
11. The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.
12. The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
13. A vector parallel to $\mathbf{c}$ but with magnitude 10.
14. A vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
15. A vector perpendicular to $\mathbf{a}$ and parallel to $\mathbf{c}$.

16. For the triangle with vertices at the points $(0, 1, 3)$, $(-1, -1, 2)$, and $(4, 2, 0)$: (a) Find the lengths of all three sides; (b) Find all three angles; (c) Find the area; (d) Find an equation for the plane containing the triangle; (e) Find the distance between $(0, 1, 3)$ and the opposite side of the triangle.

17. Find the distance between the point $(2, 1, 1)$ and the plane with equation $x - 3y + z = 9$.

18. Find an equation for the sphere with center $(3, 2, 2)$ and radius 12.

19. Find the angle between the plane with equation $4x + 3y - z = 0$ and the vector $\mathbf{i} + 5\mathbf{j}$.

20. Find a third point on the line passing through the points $(0, 2, 3)$ and $(7, 1, 1)$.

ANSWERS TO PRACTICE TEST

1. $\langle 11, 13, 4\rangle$
2. $\sqrt{(x+8)^2 + (y-2)^2 + (z+2)^2}$
3. -5
4. Impossible.
5. $\langle -6, -2, 22\rangle$
6. Impossible.
7. -4
8. $\langle -16, 7, -3\rangle$
9. $146.4°$
10. $(4x - y + z)/\sqrt{18}$
11. $\sqrt{524}$
12. 4
13. $\langle -5\sqrt{2}, 5\sqrt{2}, 0\rangle$
14. $\langle -6, -2, 22\rangle$
15. Impossible.
16. (a) $\sqrt{6}$, $\sqrt{26}$, and $\sqrt{38}$; (b) $103.9°$, $22.7°$, $53.4°$; (c) $7\sqrt{3}/2$; (d) $x - y + z = 2$; (e) $7\sqrt{3}/\sqrt{38}$.
17. $9/\sqrt{11}$
18. $(x-3)^2 + (y-2)^2 + (-2)^2 = 144$.
19. $47.0°$
20. $(-7, 3, 5)$; many other answers possible.