C2M16

Lines, Planes, and Distances in $\mathbb{R}^3$

This section deals with problems finding equations of lines and planes and distances from points to these objects. We will begin with a couple of simple examples.

Example 1: Suppose that we are given three points: $P(2,1,-1)$, $Q(-1,1,0)$, and $R(1,3,-1)$, and we wish to find an equation for the plane that contains them and the area of the triangle that they form.

Begin by subtracting one point from the other two and finding the cross product of the results. That cross product will serve as a normal vector to our plane. Every vector in the plane will have the same scalar product with that normal vector. If $\overrightarrow{N}$ is the normal vector, $\overrightarrow{X_0}$ is a specific vector in the plane, $\overrightarrow{X}$ is any vector in the plane, then $\overrightarrow{X}-\overrightarrow{X_0}$ must be orthogonal (perpendicular) to $\overrightarrow{N}$. So the basic equation for the plane becomes

$$\overrightarrow{N}\cdot(\overrightarrow{X}-\overrightarrow{X_0})=0 \qquad \text{or equivalently} \qquad \overrightarrow{N}\cdot\overrightarrow{X}=\overrightarrow{N}\cdot\overrightarrow{X_0}$$

And in Maple,

```
> restart:    with(student):    with(linalg):
> P:=vector([2,1,-1]):  Q:=vector([-1,1,0]):  R:=vector([1,3,-1]):
> QP:=evalm(P-Q);
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$$QP := [3,0,-1]$$

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> QR:=evalm(R-Q);
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$$QR := [2,2,-1]$$

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> N:=crossprod(QP,QR);
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$$N := [2,1,6]$$

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> X:=vector([x,y,z]);
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$$X := [x,y,z]$$

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> plane:=innerprod(N,X)=innerprod(N,P);
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$$plane := 2x+y+6z=-1$$

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> innerprod(N,Q);
```

$$-1$$

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> innerprod(N,R);
```

$$-1$$

We used $\overrightarrow{P}$ as the known vector in the plane. After we found the equation we took the scalar product of $\overrightarrow{N}$ with $\overrightarrow{Q}$ and $\overrightarrow{R}$ just to check and see if we got the same value. With Maple doing the calculations for us no error occurred, but when doing this very basic problem with pencil and paper it is important to check your answer.

Now let's find the area of $\triangle PQR$. The two vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ generate a parallelogram whose area is $\|\overrightarrow{QP}\times\overrightarrow{QR}\|=\|\overrightarrow{N}\|$ and the triangle is one-half of that parallelogram.

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> area:=norm(N,2)/2;;
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$$area := \frac{1}{2}\sqrt{41}$$

A line in $\mathbb{R}^3$ requires a direction and a point on the line. If $\overrightarrow{N}=\langle a,b,c\rangle$ is the direction vector, $\overrightarrow{X_0}=\langle x_0,y_0,z_0\rangle$ is a specific point on the line, t is a scalar (parameter) and $\overrightarrow{X}=\langle x,y,z\rangle$ is any point on the line, then the parametric equation of the line is

$$\overrightarrow{X}=\overrightarrow{X_0}+t\,\overrightarrow{N} \qquad \text{or equivalently} \qquad \begin{aligned} x&=x_0+at\\ y&=y_0+bt\\ z&=z_0+ct \end{aligned}$$

If we want a parametric equation of a line **normal** (perpendicular, orthogonal) to our plane above and passing through P then

```
> nline:=evalm(X=P+t*N);
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$$nline := [x,y,z]=[2+2t,1+t,-1+6t]$$

To obtain a parametric equation for the line through P in the direction of Q, we simply use $\overrightarrow{QP}$ as the direction vector.

```
> line:=evalm(X=P+t*QP);
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$$line := [x,y,z]=[2+3t,1,-1-t]$$

Let's turn our attention to the distance from a point to a plane. Suppose $ax + by + cz = d$ is an equation for the plane and P is a point. You know that the coefficients a, b, c are attitude numbers for the plane and that $\overrightarrow{N} = \langle a, b, c \rangle$ is a normal vector. Find any point X_0 which lies in the plane (X_0 satisfies the equation). Form the vector $\overrightarrow{\alpha} = \overrightarrow{P - X_0}$. The distance from the point P to the plane will be the length of the projection of $\overrightarrow{\alpha}$ on $\overrightarrow{N}$.

$$\text{distance} = \|c\overrightarrow{N}\| = \left\| \frac{\overrightarrow{\alpha} \cdot \overrightarrow{N}}{\overrightarrow{N} \cdot \overrightarrow{N}} \overrightarrow{N} \right\| = \frac{|\overrightarrow{\alpha} \cdot \overrightarrow{N}|}{\|\overrightarrow{N}\|^2} \|\overrightarrow{N}\| = \frac{|\overrightarrow{\alpha} \cdot \overrightarrow{N}|}{\|\overrightarrow{N}\|}$$

Please note the diagram for this later in the section.

Example 2: Find the dik]N44(+(ʘB+Lffi(+(ffiffi:aterE[--o'Fffi]ffi44(dL(L4']k'Nffi'o+]F4L+ffi(+(ffiffi:+ffi(+(ffiffi6N(4(k(ɓ

$$CP := [-41, -35, 13]$$

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> d:=norm(CP,2)/norm(N,2);
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$$d := \frac{1}{2}\sqrt{123}\sqrt{2}$$

Exercises (pencil and paper):

1. Find an equation for the plane that contains the points $P(2,3,-1)$, $Q(-1,1,4)$, $R(0,1,3)$ and the area of the triangle that they form. Find parametric equations for the line through P and Q.

2. Find an equation for the plane that contains the points $P(1,-1,1)$, $Q(0,2,-1)$, $R(3,3,-1)$ and the area of the triangle that they form. Find parametric equations for the line through P and Q.

3. Find the distance from the point $P(3,3,-2)$ to the plane whose equation is $2x-3y+2z=7$.

4. Find the distance from the point $Q(3,2,-1)$ to the plane whose equation is $x+4y-z=6$.

5. Find the distance from the point $P(3,3,-2)$ to the line through $Q(1,2,-1)$ and $R(3,-1,5)$ and find parametric equations for the line.

6. Find the distance from the point $P(-2,1,3)$ to the line through $Q(0,2,-1)$ and $R(-1,4,1)$ and find parametric equations for the line.

C2M16 Problems Use Maple to solve the problems 1, 3, and 5 above.