C2M9

Growth and Decay

The exponential function appears in many problems that involve physical phenomena. Newton's **Law of Cooling**, growth and decay, and the mixing of solutions come to mind as examples. In each case, the rate of change of some quantity is proportional to some aspect of the amount present. Imagine placing one steel bar at 32° Fahrenheit in a room that is 40° and another in a room that is 100°. Certainly the temperature of the second bar will change more rapidly than that of the first. Newton's **Law of Cooling** asserts that the rate of change of the surface temperature of an object is proportional to the difference between that temperature and that of the surrounding medium. So if F is the temperature of the steel bar and R_T is the room temperature, then

$$\frac{dF}{dt} = k\,(F - R_T)$$

Example: Suppose that a steel bar is cooled to -10° F and it is placed in a room that is 70° F. Ten minutes later the temperature of the bar is 25°. Find an expression for the temperature at any time t and specifically at $t = 30$. By column

$\frac{dF}{dt} = k(F - 70)$

$\frac{dF}{F-70} = k\,dt$

$\int \frac{dF}{F-70} = \int k\,dt$

$\ln|F - 70| = kt + C$ but $|F - 70| = 70 - F$

$70 - F = e^{kt+C}$ $F(0) = -10 \Rightarrow C = \ln 80$

$F = 70 - e^{kt+\ln 80} = 70 - 80e^{kt}$

$F(10) = 25 = 70 - 80\,e^{10k}$

$80\,e^{10k} = 70 - 25 = 45$

$e^{10k} = \frac{45}{80} = \frac{9}{16}$

$10k = \ln\frac{9}{16}$

$k = -.0575364$

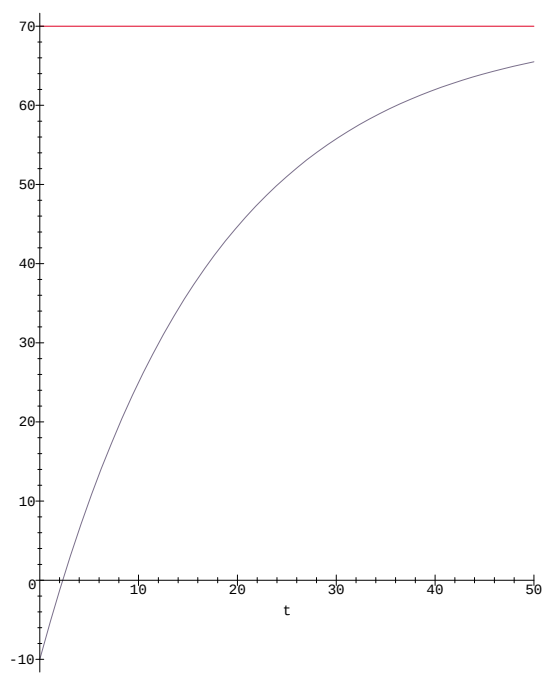
$F = 70 - 80e^{-.0575364\,t}$

which yields $F(30) = 55.7617^\circ$. The exponential term goes to zero as time increases, so let's use Maple to graph F and see how F tends to 70.

```
> with(plots):
> F:=70-80*exp(-.0575364*t);
```

$$F := 70 - 80\,\mathrm{e}^{(-.0575364\,t)}$$

```
> A:=plot(F,t=0..50):
> B:=plot(70,t=0..50):
> display(A,B);
```



Consider how the slope of the tangent line will change as a point moves from left to right along the curve. This means that the rate of change is a decreasing function, which is consistent with Newton's **Law of Cooling**.

C2M9 Problems: In problems 1 and 2, use Maple to plot the graph(s) of the given function(s) on the same axes. Remember that the exponential function e^x in Maple is `exp(x)`.

1. $f(t) = 60 + 30\,e^{-.3\,t} \quad g(t) = 60, \quad 0 \le t \le 40$
2. $h(t) = 3\,e^{-t},\ j(t) = -4\,e^{-3\,t},\ h(t) + j(t),\ 0 \le t \le 6$
3. (Pencil and paper) A steel bar is heated to 200° and placed in a room whose temperature is 60°. Thirty minutes later the bar is 110°.

   (a) Find an expression for the temperature of the bar at any time t.

   (b) At what time will the temperature of the bar be 80°?