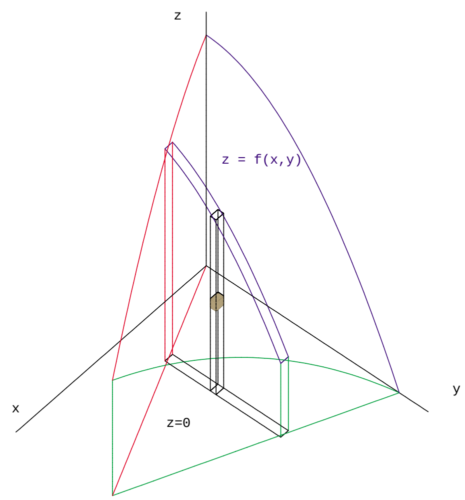
C3M11

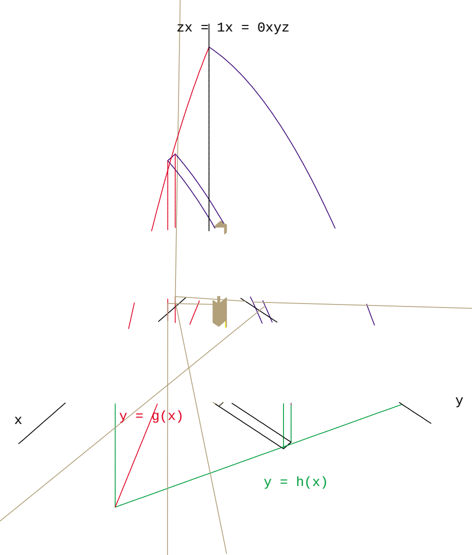
The Evaluation of Triple Integrals by Iteration

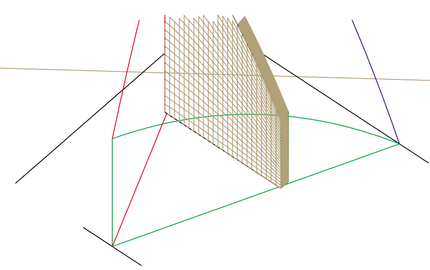
The use of triple integrals involves two separate skills, the set up and the evaluation. In order to set up a triple integral we need to understand a simple phrase, **Surface-to-Surface, Curve-to-Curve, Point-to-Point.** The first or inner integral builds "columns" by adding "cubes" between two surfaces. The second integral adds up those "columns" that are bounded between two curves to obtain "slabs". The third integral adds up the "slabs" between two points to get a volume. This simplistic view of things pre-supposes that the function $f(x, y, z) = 1$ was the integrand. The reader should realize that if the integrand is a density function, such as $\delta(x, y, z)$, whose units are "pounds per cubic foot" or "grams per cubic centimeter", then the evaluation of the integral will yield the weight of the solid in pounds or grams respectively. So triple integrals compute far more than just volume. But the visualization of computing the volume by adding up small "cubes" as a means of establishing the limits of integration is an essential part of understanding the process of integration.

In the figure that follows one sees a vertical column which may be regarded as being formed from solid cubes. If we are finding the volume of a solid, then one begins the iterative process by adding up the volume of cubes to obtain the volume of a column. The cubes range between two surfaces while two variables are held constant. Here we see change with respect to z as z ranges between $z = 0$ and $z = f(x, y)$, while x and y are held constant.



Surface - to - Surface





You saw the last two figures in **C3M8**, so they should look familiar.

There is a very important idea that we need to understand from the beginning. If the integrand is continuous over the region or solid, then the integrand has **absolutely nothing** to do with establishing the limits of integration of the triple integral. These limits are determined solely by the geometry of the solid involved. It is the relationship of the solid to the coordinate planes that either permits the integral to be set up as one integral or it forces two or more integrals to be used. There are three variables, so there are six orders of integration that are possible, including $dx\,dy\,dz$ and $dy\,dz\,dx$ and so forth. Having said that, the process of successive anti-differentiation may range from easy to impossible when different possible orders are considered. We begin with a picture of a solid figure. After the first integral we look at that solid so that our line of sight is parallel to the columns we just built. We now see a two-dimensional view of the solid with the ends of the columns appearing as squares. We have reduced a triple integral to a double integral, which we know how to evaluate.

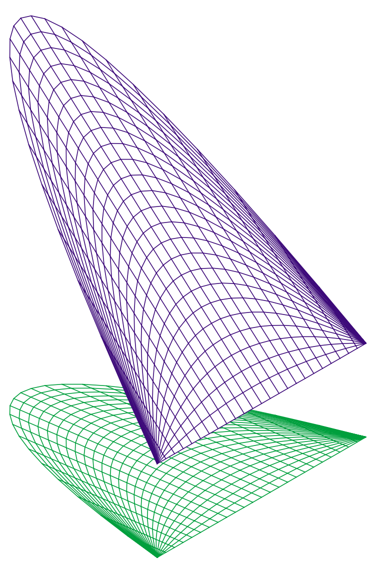
In a previous chapter we learned how to find partial derivatives. Essentially this involved differentiating with respect to one variable while the other variables were held constant. To evaluate a triple integral in iterated form we anti-differentiate with respect to one variable while the others are held constant, evaluate at the (variable) endpoints, and repeat this process with respect to a different variable. For all practical purposes we are "partially integrating" instead of "partially differentiating. If we begin by anti-differentiating with respect to z, then the integration takes place between two functions, say $z = g(x, y)$ and $z = h(x, y)$. It would look like

$$\int_{x=a}^{x=b} \int_{y=r(x)}^{y=s(x)} \left[F(x,y,z)\Big|_{z=g(x,y)}^{z=h(x,y)} \right] dy\,dx$$

after the first anti-differentiation. The curves $y = r(x)$ and $y = s(x)$ were seen in outline form as we looked down the z-axis after integrating with respect to z. It is important to note that after the substitution of $z = g(x, y)$ and $z = h(x, y)$ into $F(x, y, z)$, the variable z disappears and a double integral in terms of x and y remains.

Example 1 Suppose a solid is bounded by $y + z = 4$, $z = 0$, $y = x^2$, $y = 3$. Find the volume of the solid by using an iterated integral. The first or inner integral will have a roof $y + z = 4$ and a floor $z = 0$ as bounds. Because "floor" is a protected word in Maple, we will use "floor1" instead.

```
> with(student):with(plots):
> roof:=plot3d([x,y,4-y],x=-sqrt(3)..sqrt(3),y=x^2..3,color=blue):
> floor1:=plot3d([x,y,0], x=-sqrt(3)..sqrt(3),y=x^2..3,color=green):
> display(roof,floor1);
```

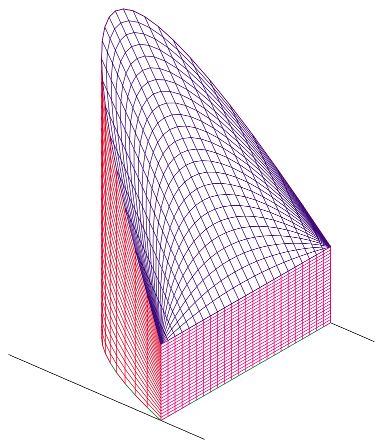


The integral for this part is

$$\int_{z=0}^{z=4-y} 1\,dz = 4 - y$$

Now we add the curved side $y = x^2$ and the vertical plane $y = 3$ as well as two lines which show where $x = -\sqrt{3}$ and $x = \sqrt{3}$..

```
> cside:=plot3d([x,x^2,z], x=-sqrt(3)..sqrt(3),z=0..4-x^2,color=red):
> vside:=plot3d([x,3,z], x=-sqrt(3)..sqrt(3),z=0..1,color=magenta):
> L1:=spacecurve([-sqrt(3),y,0],y=-.5..4,color=black):
> L2:=spacecurve([sqrt(3),y,0],y=-.5..4,color=black):
> display(roof,floor1,cside,vside,L1,L2);
```



Now we integrate between the curves $y = x^2$ and $y = 3$.

$$\int_{y=x^2}^{y=3} 4 - y\,dy = 4y - \frac{y^2}{2}\Big|_{y=x^2}^{y=3} = \left(12 - \frac{9}{2}\right) - \left(4x^2 - \frac{x^4}{2}\right) = \frac{15}{2} - 4x^2 + \frac{x^4}{2}$$

It remains to integrate this last result between the x values of $\pm\sqrt{3}$.

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{15}{2} - 4x^2 + \frac{x^4}{2}\,dx = \frac{44}{5}\sqrt{3}$$

The triple integral that accomplishes the same thing is

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{y=x^2}^{y=3} \int_{z=0}^{z=4-y} 1\,dz\,dy\,dx$$

The Maple integral would be

```
> Q:=Tripleint(1,z=0..4-y,y=x^2..3,x=-sqrt(3)..sqrt(3));
```

$$Q := \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y=x^2}^{y=3} \int_{z=0}^{z=4-y} 1\,dz\,dy\,dx$$

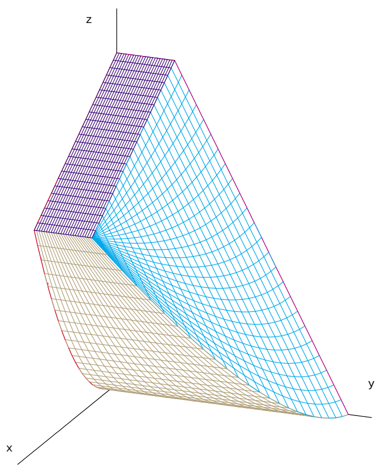
```
> value(Q);
```

$$\frac{44}{5}\sqrt{3}$$

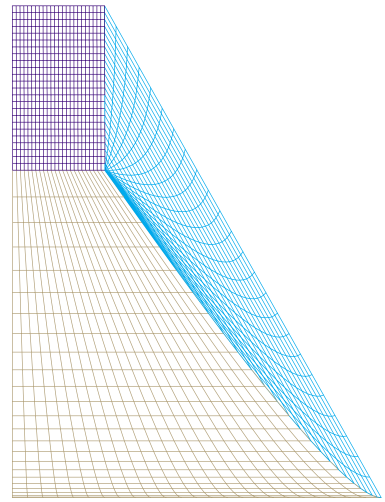
Compare the Maple syntax for the triple integral with that for drawing 'roof' and 'floor1'. The order of the limits in '`tripleint`' determines the order of the variables to be integrated. Now let's consider an example where z is not the first variable to be integrated. In fact, let's really analyze how we must select the order of integration.

Example 2 Set up an iterated triple integral to find $\iiint_S x\,dV$ where S is bounded by $z = 2x^2$, $x + y + z = 4$, $x + z = 3$, $x = 0$, and $y = 0$.

We begin with a view of the solid to show the three surfaces which do not lie in a coordinate plane and the three views down each of the axes.

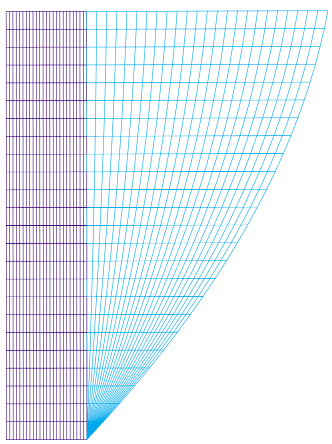


the solid

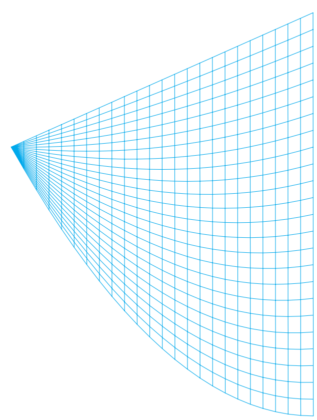


down the x-axis

From our view of the solid, we realize that we may not just assume that z is first. Looking down the x-axis we see three different surfaces. A light beam from behind the figure would always enter through the single surface where $x = 0$. But as it exits towards us, it could leave through *three* different surfaces, which denies the choice of integrating with respect to x first. Then we look down the z-axis from above the figure.



down the z-axis



down the y-axis

We see that a light beam entering the solid from below the figure would always enter through the parabolic surface $z = 2x^2$. That beam may exit through either of two planes, $x + z = 3$ or $x + y + z = 4$. Because there are two exit surfaces we should not choose z as the first variable of integration - unless we want to break the integral up into two distinct integrals over solids which abut. Last, and best, we look down the y-axis and realize that a light from behind would enter in the plane $y = 0$ and exit towards us through the plane $x + y + z = 4$. Ahah! Our first choice should (must) be to integrate first with respect to y and build columns from left to right (when looking at our first view of the solid) which are parallel to the y-axis. We begin to build our integral and show the variables in the limits at first:

$$\text{First step} \qquad \int\int\int_{y=0}^{y=4-x-y} x\, dy\, dA \qquad\qquad dA = dx\, dz \ \text{ or } \ dA = dz\, dx$$

Now, and this is important, return to the view down the y-axis shown above and look at this as if it were two-dimensional and is the region for a double integral. In our first integral we went ‘surface-to-surface’, and now we must go ‘curve-to-curve’ as before. Integrating right to left (positive direction) would not work because the light beam would exit through two different curves. Our only choice is vertical, and the light beam test allows the light to enter through the parabola and exit through the line when the light is below.

Second step $\displaystyle \int\int_{z=2x^2}^{z=3-x}\int_{y=0}^{y=4-x-z} x\,dy\,dz\,dx \implies \int_{x=0}^{x=1}\int_{z=2x^2}^{z=3-x}\int_{y=0}^{y=4-x-z} x\,dy\,dz\,dx$ Third step

Now consider the Maple syntax for this triple integral and the graphics which produced the solid as seen above when displayed, except for the coordinate axes.

```
> with(student):    with(plots):
> A2:=Tripleint(x, y=0..4-x-z, z=2*x^2..3-x, x=0..1);
```

$$A2 := \int_0^1\int_{2x^2}^{3-x}\int_0^{4-x-z} x\,dy\,dz\,dx$$

```
> value(A2);
```

$$\frac{51}{40}$$

```
> bottom:=plot3d([x,y,2*x^2],x=0..1,y=0..4-x-2*x^2,color=blue):
> top:=plot3d([x,y,3-x],x=0..1,y=0..1,color=red):
> slant:=plot3d([x,4-x-z,z],x=0..1,z=2*x^2..3-x,color=cyan):
> left:=plot3d([x,0,z],x=0..1,z=2*x^2..3-x,color=magenta):
> back:=plot3d([0,y,z],z=0..3,y=0..4-z,color=green):
> display(bottom,top,slant,left,back);
```

C3M11 Problems

Evaluate the triple integrals in 1, 2, and 3 by pencil and paper and by Maple to check your answers. Use Maple to plot the solid figure which is the domain of the integral. Hint: Insert $x =, y =, z =$ appropriately in the limits of the integrals before you begin, to help find equations for the surfaces and curves. Ignore the integrand when sketching the solid.

1. $A = \displaystyle\int_{-1}^{2}\int_0^{\pi}\int_1^4 yz\cos(xy)\,dz\,dx\,dy$
2. $B = \displaystyle\int_0^2\int_0^{3-x}\int_0^{6-x-y} x\,dz\,dy\,dx$
3. $C = \displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-2y} x\,dz\,dy\,dx$

In the remaining problems, use Maple to sketch the solid and to find the volume by evaluating the triple integral .

4. The solid is bounded by $x-y+z=2,\ x=z^2,\ x+z=2,\ y=0,\ x=0.$
5. The solid is bounded by $z=2-x^2,\ x=z,\ x+y+z=3,\ x=0,\ y=0.$
6. The solid is bounded by $z-y=2,\ y=2,\ y+z=2,\ x-y^2=2,\ x=0.$