C3M12a

Evaluating Integrals using Cylindrical Coordinates

Let's begin with an integral in rectangular coordinates, plot the solid which is the domain of the integral, plot the same solid using cylindrical coordinates, and then transform the integral into one using cylindrical coordinates. A simple first step is to include the variable of integration in the limits of the integral, which allows one to see the appropriate equations that define the surfaces. Immediately below the integral we will list the necessary packages and the Maple command that would produce this integral.

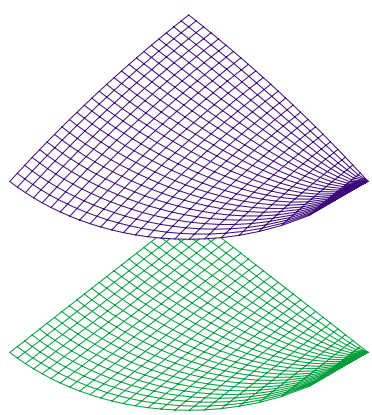
Example: $$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=0}^{z=\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy$$

```
> with(student):with(plots):
> A:=Tripleint(x*z,z=0..sqrt(4-x^2-y^2),x=0..sqrt(1-y^2),y=0..1);
> valA:=value(A);
```

$$valA := \frac{17}{30}$$

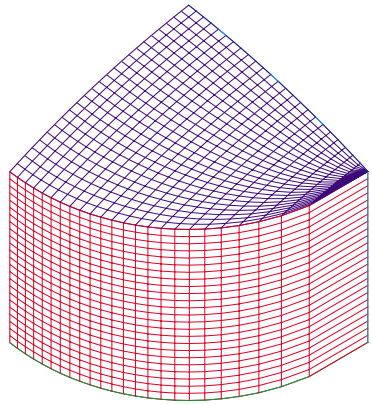
We will use `plot3d` in parametric form, `[x,y,z]`, with one of the variables eliminated so that just two are allowed to vary and that variable does not appear. The inner integral varies between two surfaces which, for intuitive reasons, we will designate as $floor1$ and $roof1$. Pay close attention to the similarities between the syntax for the triple integral and for these plots.

```
> floor1:=plot3d([x,y,0],x=0..sqrt(1-y^2),y=0..1,color=green):
> roof1:=plot3d([x,y,sqrt(4-x^2-y^2),x=0..sqrt(1-y^2),y=0..1,color=blue):
> display(floor1,roof1);
```



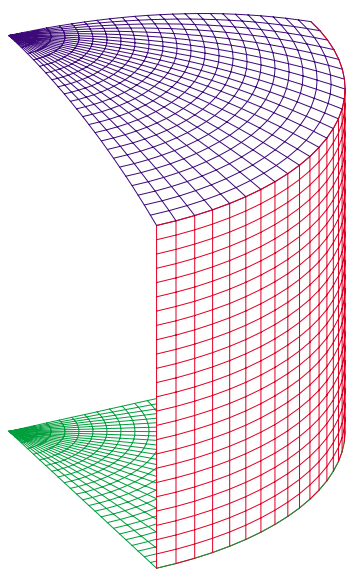
Now let's add the cylindrical wall, $cwall1$, and the two flat vertical walls.

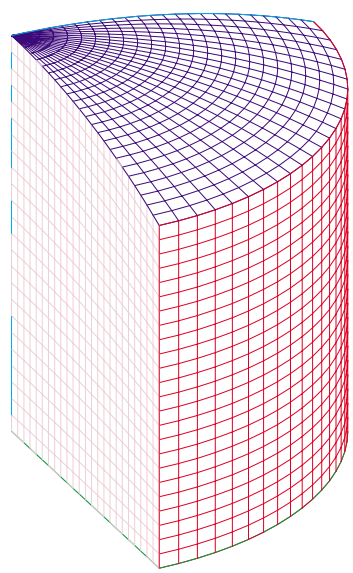
```
> cwall1:=plot3d([sqrt(1-y^2),y,z],z=0..sqrt(3),y=0..1,color=red):
> vwall1:=plot3d([0,y,z],z=0..sqrt(4-y^2),y=0..1,color=cyan):
> vwall2:=plot3d([x,0,z],z=0..sqrt(4-x^2),x=0..1,color=plum):
> display(floor1,roof1,cwall1,vwall1,vwall2);
```



We must see how to sketch this same solid in cylindrical coordinates. We know at this point that the solid is a quarter of a circular cylinder, cut off on the bottom by a horizontal plane $(z = 0)$ and on the top by a sphere $(x^2 + y^2 + z^2 = 4)$. All of this takes place in the first octant which is where $x \geq 0$ and $y \geq 0$. You will find that this solid was discussed in the section on plotting using `cylinderplot`. We will use `cylinderplot`, which is a 3d-plotter using coordinates $[r, \theta, z]$ in parametric form. It may also be used as `cylinderplot(f(r,t),r=g(t)..h(t),t=a..b)` where the surface is defined by $z = f(r, t)$ in cylindrical coordinates. It is easy to see that we have used t instead of θ here. The figure below has been rotated to provide a better viewing angle. In cylindrical coordinates we have the following with its output on the left below.

```
> roof2:=cylinderplot([r,t,sqrt(4-r^2),r=0..1,t=0..Pi/2,orientation=[-26,69],color=blue):
> floor2:=cylinderplot([r,t,0],r=0..1,t=0..Pi/2.color=green):
> cwall2:=cylinderplot([1,t,z],t=0..Pi/2,z=0..sqrt(3),color=red):
> display(floor2,roof2,cwall2);
```





Adding two more walls gives the figure on the right above as the output of the next commands.

```
> vwall3:=cylinderplot([r,Pi/2,z],z=0..sqrt(4-r^2),r=0..1,color=plum):
> vwall4:=cylinderplot(r,0,z],z=0..sqrt(4-r^2),r=0..1,color=cyan):
> display(floor2,roof2,cwall2,vwall3,vwall4);
```

We are almost ready to set up our triple integral but first must determine the integrand. This is the

one part of the process that is dealt with as a simple direct substitution. In cylindrical coordinates we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Our integrand in rectangular coordinates is $f(x, y, z) = xz$, so in Maple

```
> f:=(x,y,z)->x*z;
```

$$f := (x, y, z) \rightarrow xz$$

```
> F:=simplify(f(r*cos(t),r*sin(t),z));
```

$$F := r\,\cos(t)\,z$$

IMPORTANT: Note the use of **F*r** for the integrand in our cylindical coordinate integral!

```
> B:=Tripleint(F*r,z=0..sqrt(4-r^2),r=0..1,t=0..Pi/2);
```

$$B := \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{4-r^2}} r^2\,\cos(t)\,z\,dz\,dr\,dt$$

```
> Bval:=value(B);
```

$$Bval := \frac{17}{30}$$

With the limits displayed fully, we have

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}}\int_{z=0}^{\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\int_{z=0}^{\sqrt{4-r^2}} r^2\cos(\theta)z\,dz\,dr\,d\theta\,.$$

C3M12a Problems

Use Maple to find the value of an equivalent integral in cylindrical coordinates. Use `cylinderplot` to plot the solid which represents the domain of the integral.

1. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

2. $\displaystyle\int_{0}^{2}\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}\int_{0}^{x^2+y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$