C3M12b

Evaluating Integrals Using Spherical Coordinates

We will begin with plotting using '`sphereplot`' and then address the integrals. In parametric form `sphereplot` uses $[\rho, \theta, \phi]$, where one of the three is in terms of the other two or is a constant. Remember that ϕ is measured down from the "north pole" or the usual z-axis. To understand the relationships between the coordinate systems it is good to start with the r of polar or cylindrical coordinates and recall that

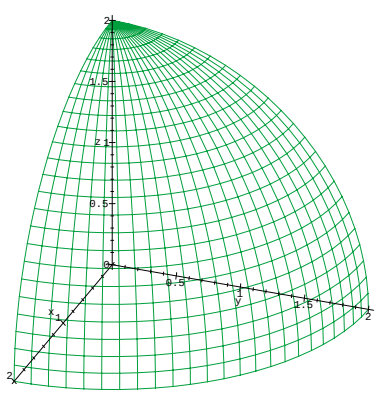
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Building on this, use the fact that $r = \rho\sin\phi$ to obtain for $[\rho, \theta, \phi]$

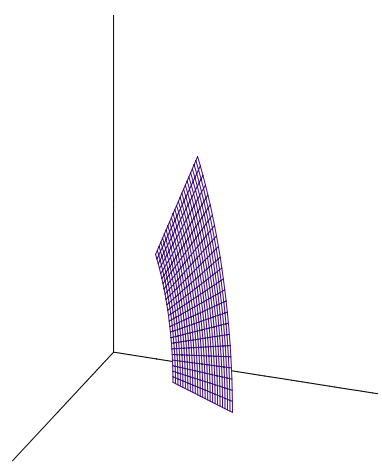
$$\begin{aligned} x &= (r)\cos\theta \\ y &= (r)\sin\theta \\ z &= \rho\cos\phi \end{aligned} \qquad = \qquad \begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

It helps if we understand what it means for each of the variables to be held constant while the other two vary. In the first of three simple plots we hold ρ constant. This should produce some portion of a sphere. In order to have the axes appear at the origin we have included a graph in white which we do not list here. It is not necessary to use the Greek names of the variables, but for demonstration purposes it is easiest. The output is on the left.

```
> sphereplot([2,theta,phi],theta=0..Pi/2,phi=0..Pi/2,color=green);
```



ρ is constant



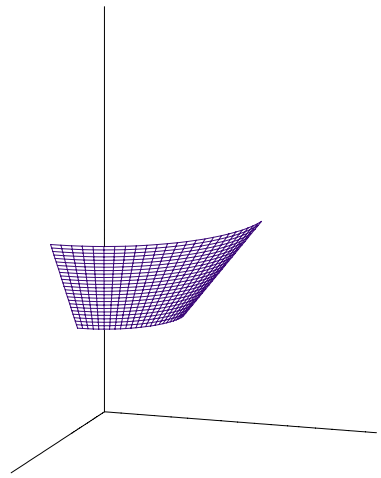
θ is constant

With the output above on the right, hold θ constant. The surface should be in the vertical plane $\theta = \pi/3$.

```
> sphereplot([rho,Pi/3,phi],rho=1..2,phi=Pi/4..Pi/2,color=blue);
```

Now hold ϕ constant. This restricts the surface to a portion of some cone.

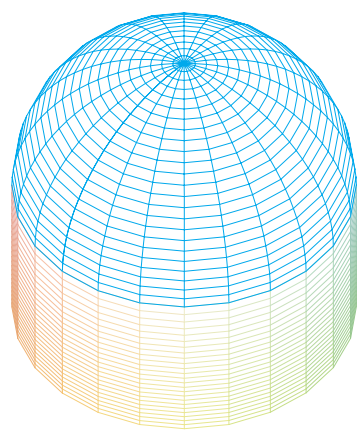
```
> sphereplot([rho,theta,Pi/3],rho=1..2,theta=0..Pi/2,color=blue);
```



ϕ is constant

Recall that in polar coordinates $r = a\sec\theta$ produced a vertical line, $x = a$. And, $r = a\csc\theta$ produced a horizontal line, $y = a$. The equation $r = a\cos\theta$ yielded a circle, but here, the angle ϕ is measured from the z-axis, so $\rho = a\cos\phi$ would produce a circle on the z-axis for *every* θ. This results in a sphere tangent to the xy-plane. Let's combine that with an equation of the form $\rho = a\csc\phi$, which is a vertical cylinder of radius a..

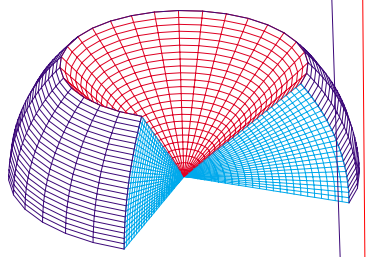
```
> A:=sphereplot([4*cos(phi),theta,phi],theta=0..2*Pi,phi=0..Pi/4,color=cyan):
> B:=sphereplot([2*csc(phi),theta,phi],theta=0..2*Pi,phi=Pi/4..Pi/2):
> display(A,B);
```



Now we turn to the integrals and begin with an example.

Example 1 Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 9$, above the xy-plane, below the upper portion of the cone $x^2 + y^2 = z^2$, and excluding the first octant. This solid would require more than one integral in rectangular coordinates, while in spherical it is quite straightforward. Remember that in spherical coordinates $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

```
> with(student):  with(plots):
> sph1:=sphereplot([3,theta,phi],theta=Pi/2..2*Pi,phi=Pi/4..Pi/2,color=blue):
> cone1:=sphereplot([rho,theta,Pi/4], theta=Pi/2..2*Pi,rho=0..3,color=red):
> side1:=sphereplot([rho,Pi/2,phi],rho=0..3,phi=Pi/4..Pi/2,color=cyan):
> side2:=sphereplot([rho,0,phi],rho=0..3,phi=Pi/4..Pi/2,color=cyan):
> display(sph1,cone1,side1,side2);
```



```
> V1:=Tripleint(rho^2*sin(phi),rho=0..3,phi=Pi/4..Pi/2,theta=Pi/2..2*Pi);
```

$$V1 := \int_{\pi/2}^{2\pi} \int_{\pi/4}^{\pi/2} \int_{0}^{3} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

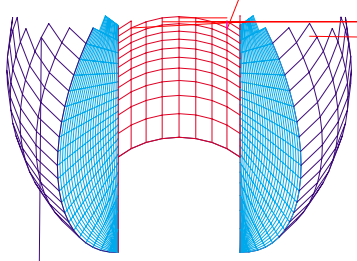
```
> valV1:=value(V1);
```

$$valV1 := \frac{27}{4}\pi\sqrt{2}$$

Example 2 Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4$ that is **outside** the cylinder $x^2 + y^2 = 1$.

A three-quarter view of this solid results from:

```
> sph2:=sphereplot([2,theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6,color=blue):
> cyl2:=sphereplot([csc(phi),theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6,color=red):
> side3:=sphereplot([rho,0,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6,color=cyan):
> side4:=sphereplot([rho,Pi/2,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6,color=cyan):
> display(sph2,cyl2,side3,side4);
```



$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$