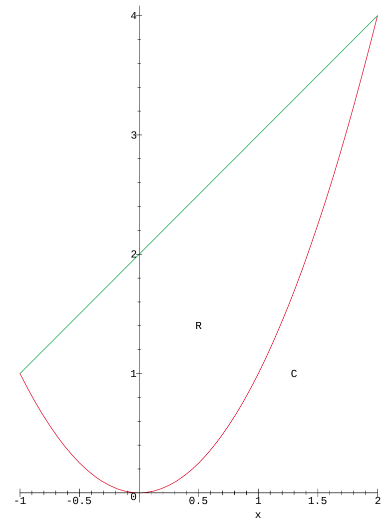
C3M15

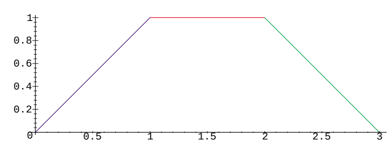
Notes on Line Integrals - Green's Theorem

GREEN'S THEOREM: Let R be a simple region in the xy-plane with a piecewise smooth boundary C that is oriented counterclockwise. Let $\vec{F}$ be a vector field with all relevant components and their partial derivatives continuous on an open region containing R. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Big)\,dA$$



Example 1 Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y) = \langle 12x^2 \sin y + 3xy^2,\, 4x^3 \cos y + 6x^2 y \rangle$ and the polygonal path $(0,0) \to (3,0) \to (2,1) \to (1,1) \to (0,0)$.



We form the integrand for the double integral in Green's Theorem.

$$\frac{\partial N}{\partial x} = 12x^2 \cos y + 12xy \quad \text{and} \quad \frac{\partial M}{\partial y} = 12x^2 \cos y + 6xy \quad \Longrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy$$

It follows from Green's Theorem that

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \Big)\,dA \\ &= \int_0^1 \int_y^{3-y} 6xy\,dx\,dy \\ &= \int_0^1 3x^2 y \Big|_y^{3-y}\,dy = \int_0^1 3(3-y)^2 y - 3y^3\,dy \\ &= \int_0^1 27y - 18y^2\,dy = \frac{27}{2}y^2 - 6y^3 \Big|_0^1 \\ &= \frac{15}{2} \end{aligned}$$

Example 2 (Example 2 of C3M14 revisited) The vector-valued function was $\vec{F}(x,y) = \langle -y, x \rangle$. Please refer back for a diagram of the region. With $M(x,y) = -y$ and $N(x,y) = x$,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$$

So we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 2\, dA = (2)\text{area}(R) = (2)\left(\frac{\pi}{2} + 2\right) = \pi + 4$$

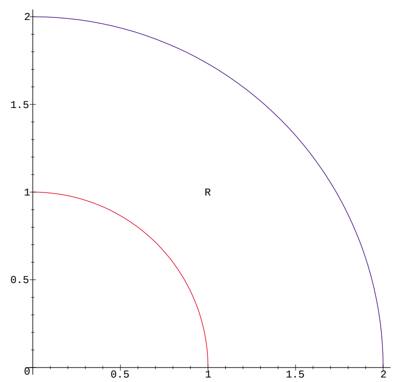
The computation of the area was easy - one half of a disk of radius 1 and one half of a square of side 2.

Example 3 The region R is that portion of the first quadrant between the circles of radius 1 and 2 centered at the origin. The vector-valued function is

$$\vec{F}(x, y) = \langle 4 + e^{\sqrt{x}}, \sin(y) + 3x^2 \rangle$$

and the objective is to evaluate $\oint_C \vec{F} \cdot d\vec{r}$ if C is the boundary of R traversed in a counterclockwise manner. It is not easy to evaluate the line integral around this path, so our approach is to use Green's theorem. We also observe that polar coordinates provides the simplest double integral. First, we will display the region.

```
> with(student):    with(plots):
> Q1:=polarplot([1,t,t=0..Pi/2]):    Q2:=polarplot([2,t,t=0..Pi/2]):
> T:=textplot([1,1,`R`])
> display(Q1,Q2,T);
```




```
> F:=(x,y)->[4+exp(sqrt(x)),sin(y)+3*x^2];
```

$$F := (x, y) \to [4 + e^{\sqrt{x}}, \sin(y) + 3x^2]$$

```
> Nx:=diff(F(x,y)[2],x);
```

$$Nx := 6x$$

```
> My:=diff(F(x,y)[1],y);
```

$$My := 0$$

```
> grand:=Nx-My;
```

$$grand := 6x$$

```
> grand:=subs(x=r*cos(t),grand);
```

$$grand := 6\, r\, \cos(t)$$

```
> Ans:=Doubleint(grand*r,r=1..2,t=0..Pi/2);
```

$$Ans := \int_0^{\pi/2} \int_1^2 6r^2 \cos(t)\, dr\, dt$$

```
> Greenans:=value(Ans);
```

$$Greenans := 14$$

Example 4 If $\vec{F}(x, y) = \langle -y/2, x/2 \rangle$, then $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = \dfrac{1}{2} - \left(-\dfrac{1}{2}\right) = 1$ and

$$\iint_R 1\, dA = \text{area}(R)$$

gives us a way of using line integrals to determine the area of a region. If the objective is to find the area of a region whose boundary is reasonable to parameterize, then **choose** the vector-valued function $\vec{F}(x, y) = \langle -y/2, x/2 \rangle$ and apply Green's Theorem by evaluating the resulting line integral.

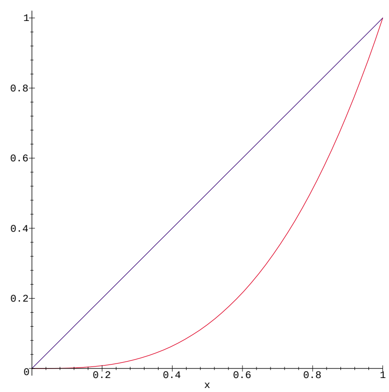
Suppose we find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. We parametrize C in the counterclockwise direction by $\vec{\alpha}(t) = \langle a\cos t, b\sin t\rangle$ for $0 \le t \le 2\pi$. With $\vec{F}(x,y) = \langle -y/2, x/2\rangle$ we have

$$\begin{aligned}
\vec{F}\big(\vec{\alpha}(t)\big) &= \langle -\frac{b}{2}\sin t,\ \frac{a}{2}\cos t\rangle \\
\vec{\alpha}\,'(t) &= \langle -a\sin t,\ b\cos t\rangle \\
\vec{F}\big(\vec{\alpha}(t)\big)\cdot\vec{\alpha}\,'(t) &= \frac{ab}{2}\sin^2 t + \frac{ab}{2}\cos^2 t = \frac{ab}{2} \\
\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \frac{ab}{2}\,dt \\
&= \pi ab
\end{aligned}$$

Thus, the area of the ellipse is πab. What happens if $a = b$?

Example 5 Verify Green's Theorem for $\vec{F}(x,y) = \langle y^3, x^3 + 3xy^2\rangle$ and the region R which lies between $y = x$ and $y = x^3$.

```
> with(student):    with(plots):    with(linalg):
> plot([x^3,x]x=0..1);
```




```
> M:=(x,y)->y^3;  N:=(x,y)->x^3+3*x*y^2;
```

$$M := (x,y) \to y^3$$
$$N := (x,y) \to x^3 + 3xy^2$$

```
> F:=(x,y)->[M(x,y),N(x,y)];
```

$$F := (x,y) \to [M(x,y), N(x,y)]$$

Let's begin by evaluating the line integral along the cubic path.

```
> alpha:=[t,t^3];
```

$$\alpha := [t, t^3]$$

```
> Falpha:=F(op(alpha));
```

$$Falpha := [t^9, t^3 + 3t^7]$$

```
> alphaprime:=diff(alpha,t);
```

$$alphaprime := [1, 3t^2]$$

```
> grand1:=innerprod(Falpha,alphaprime);
```

$$grand1 := t^9 + 3(t^3 + 3t^7)t^2$$

```
> Lint1:=Int(grand1,t=0..1);
```

$$Lint1 := \int_0^1 t^9 + 3(t^3 + 3t^7)t^2\,dt$$

```
> V1:=value(Lint1);
```

```
> Fbeta:=F(op(beta));
```

$$Fbeta := [(1-t)^3, 4(1-t)^3]$$

```
> betaprime:=diff(beta,t);
```

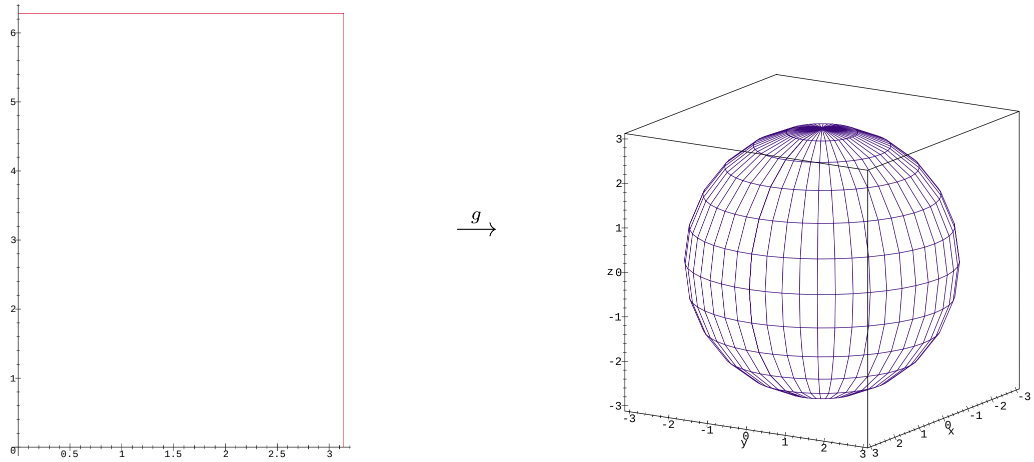
$$betaprime := [-1, -1]$$

```
> grand2:=innerprod(Fbeta,betaprime);
```

$$grand2 := 5(-1+t)^3$$

```
> Lint2:=Int(grand2,t=0..1);
```

$$Lint2 := \int_0^1$$



$$g(\theta,\varphi) = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{pmatrix} \qquad \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

And

$$\frac{\partial g}{\partial \theta} = \begin{pmatrix} -\rho\sin\varphi\sin\theta \\ \rho\sin\varphi\cos\theta \\ 0 \end{pmatrix} \qquad \frac{\partial g}{\partial \varphi} = \begin{pmatrix} \rho\cos\varphi\cos\theta \\ \rho\cos\varphi\sin\theta \\ -\rho\sin\varphi \end{pmatrix}$$

The cross product (note the order) is

$$\begin{aligned} \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} &= \begin{vmatrix} \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} \\ &= \left\langle \rho^2\sin^2\varphi\cos\theta,\ \rho^2\sin^2\varphi\sin\theta,\ \rho^2\sin\varphi\cos\varphi\cos^2\varphi + \rho^2\sin\varphi\cos\varphi\sin^2\theta \right\rangle \\ &= \left\langle \rho^2\sin^2\varphi\cos\theta,\ \rho^2\sin^2\varphi\sin\theta,\ \rho^2\sin\varphi\cos\varphi \right\rangle \end{aligned}$$

So we have

$$\begin{aligned} \left\| \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} \right\| &= \rho^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi} \\ &= \rho^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} \\ &= \rho^2|\sin\varphi| = \rho^2\sin\varphi, \qquad (0 \le \varphi \le \pi) \\ d\sigma &= \rho^2\sin\varphi\, d\varphi\, d\theta \end{aligned}$$

Does this look familiar? It should.

If we wish to find the surface area of this sphere we compute

$$\begin{aligned} \iint_S \|dS\| &= \iint_S d\sigma \\ &= \int_0^{2\pi}\int_0^{\pi} \rho^2\sin\varphi\, d\varphi\, d\theta \\ &= \int_0^{2\pi} -\rho^2\cos\varphi\Big|_0^{\pi} \\ &= \int_0^{2\pi} 2\rho^2\, d\theta = 2\rho^2\theta\Big|_0^{2\pi} \\ &= 4\pi\rho^2 \end{aligned}$$

Now we are ready to introduce

SURFACE INTEGRALS

Suppose we wish to compute the average temperature on a surface. We would need a function that yielded the temperature at each point on the surface. Then we would integrate that function over the surface and divide this result by the surface area to get the average. This approach was introduced in the single variable case in Calculus I or II.

If f is a real-valued function defined on a surface S parametrized by g on D, then the integral of f over S is defined by

$$\iint_S f\|dS\| \equiv \iint_S f\,d\sigma = \iint_D f\big(g(u,v)\big)\left\|\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v}\right\| du\,dv$$

PARALLEL Compare the working definitions for line integrals and surface integrals

$$\int_C f\,ds = \int_a^b f\big(\vec{\alpha}(t)\big)\|\vec{\alpha}\,'(t)\|\,dt$$

$$\iint_S f\,d\sigma = \iint_D f\big(g(u,v)\big)\left\|\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v}\right\| du\,dv$$

EXAMPLE 2 Let's compute the surface integral $\iint_S e^z\,d\sigma$ where S is the sphere of radius a. We may let $d\sigma = a^2\sin\varphi\,d\varphi\,d\theta$ from what we found in the preceding example. Thus

$$\begin{aligned}\iint_S e^z\,d\sigma &= \int_0^{2\pi}\int_0^{\pi} e^{a\cos\varphi}a^2\sin\varphi\,d\varphi\,d\theta \qquad \begin{pmatrix} u = a\cos\varphi \\ du = -a\sin\varphi\,d\varphi\end{pmatrix}\\ &= a\int_0^{2\pi} -e^{a\cos\varphi}\Big|_{\varphi=0}^{\varphi=\pi} d\theta\\ &= a\int_0^{2\pi} -e^{-a}+e^a\,d\theta\\ &= 2\pi a\big(e^a - e^{-a}\big)\end{aligned}$$

THE UNIT NORMAL VECTOR $\vec{n}(u,v)$. Earlier we found $\dfrac{\partial g}{\partial u}\times\dfrac{\partial g}{\partial v}$, the fundamental cross product, and mentioned that it is a vector that is orthogonal or normal to the surface S. But $\dfrac{\partial g}{\partial v}\times\dfrac{\partial g}{\partial u} = -\dfrac{\partial g}{\partial u}\times\dfrac{\partial g}{\partial v}$ is also normal to the surface and is opposite in direction. At this point we must read about oriented surfaces or listen to what the instructor says about them. Remember, the Möbius strip is <u>not</u> orientable while the surface of a sphere is orientable. Consequently, we may refer to the "outward" normal, which means that a minus sign may be involved implicitly. In any case, we write

$$\vec{n} = \frac{\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v}}{\left\|\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v}\right\|}$$

which is a <u>unit</u> normal to the surface.

As in RECALL, we assume that a vector field $\vec{F}$ is defined on the surface S parametrized by g on D. In C3M10 we defined $d\sigma = \left\|\dfrac{\partial g}{\partial u}\times\dfrac{\partial g}{\partial v}\right\| du\,dv$. A parallelogram was alluded to then, but was not specified. Consider the parallelogram generated by the two vectors, $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$, at the point $g(u,v)$ on the surface S. Each is tangent to S and so is this parallelogram that they generate. By multiplying the first vector by du and the second by dv, a smaller parallelogram is determined and the length of their cross product is the area of this parallelogram. This length is regarded as an element of area, just like $dx\,dy$ was when double integrals were introduced.

$$
\begin{aligned}
dS &= \frac{\partial g}{\partial u}\,du \times \frac{\partial g}{\partial v}\,dv \\
&= \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\,du\,dv \\
&= \vec{n}\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|\,du\,dv \\
&= \vec{n}\,d\sigma \\
\text{and} \qquad d\sigma &= \|dS\|
\end{aligned}
$$

Suppose that $\vec{F}$ is a vector-valued function and that a path Γ and a surface Σ are in its domain. If Γ is parameterized by $\vec{\alpha}(t)$ and $\vec{T} = \vec{\alpha}\,(t)/\|\vec{\alpha}\,(t)\|$ is the unit tangent, then $\vec{T}\,ds$ and ds for line integrals are analogous to dS and $d\sigma$, respectively, for surface integrals. While $\vec{F}\cdot\vec{T}$ provided the component of $\vec{F}$ tangential to the path, $\vec{F}\cdot\vec{n}$ will provide the normal component of $\vec{F}$ at the surface. We may interpret this last entity as rate of flow outward at the surface if the function $\vec{F}$ represents fluid flow. For this reason we define the

Flux Integral of $\vec{F}$ over S

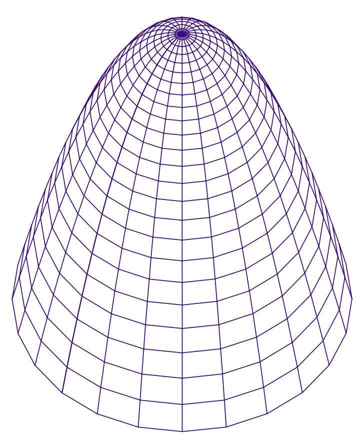
$$
\begin{aligned}
\iint_S \vec{F}\cdot\vec{n}\,d\sigma &= \iint_D \vec{F}\big(g(u,v)\big)\cdot\vec{n}\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|\,du\,dv \\
&= \iint_D \vec{F}\big(g(u,v)\big)\cdot\left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right)\,du\,dv
\end{aligned}
$$

which simplifies because of $\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|$ in both numerator and denominator. Actually, we will be more concerned about integrals of this type when the vector-valued function is the CURL of $\vec{F}$, i.e. $\nabla\times\vec{F}$.

EXAMPLE 3 (to be revisited)

Integrate the normal component of the curl of $\vec{F}(x,y,z) = \langle y^2, x, -xz\rangle$ over the surface $S = \{(x,y,z) : z = 9 - x^2 - y^2, z \ge 0\}$. That is, evaluate

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$



First,

$$\nabla\times\vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & -xz \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \langle 0, z, 1-2y\rangle$$

We parametrize S as a function of r and θ over the rectangle $D = \{(r,\theta) : 0 \le r \le 3, 0 \le \theta \le 2\pi\}$.

$$g(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 9-r^2 \end{pmatrix} \qquad \frac{\partial g}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ -2r \end{pmatrix} \qquad \frac{\partial g}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

And we compute the other components of the integral, beginning with the curl of $\vec{F}$ evaluated at $g(r,\theta)$.

$$\begin{aligned}
(\nabla \times \vec{F})\big(g(r,\theta)\big) &= \langle 0, 9-r^2, 1-2r\sin\theta\rangle \\
\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} &= \begin{vmatrix} \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle \\
(\nabla \times \vec{F})\big(g(r,\theta)\big) \cdot \left(\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta}\right) &= (9-r^2)(2r^2\sin\theta) + r - 2r^2\sin\theta \\
&= 16r^2\sin\theta - 2r^4\sin\theta + r
\end{aligned}$$

Putting this all together we have

$$\begin{aligned}
\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma &= \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} (16r^2 - 2r^4)\sin\theta + r\, d\theta\, dr \\
&= \int_{r=0}^{r=3} -(16r^2 - 2r^4)\cos\theta + r\theta \Big|_{\theta=0}^{\theta=2\pi} dr \\
&= \int_0^3 2\pi r\, dr = \pi r^2 \Big|_0^3 \\
&= 9\pi
\end{aligned}$$