

The Divergence Theorem

THE DIVERGENCE THEOREM: Let Q be a simple solid region whose boundary surface S is oriented by the unit normal \vec{n} directed outward from Q , and let \vec{F} be a vector field whose component functions have continuous partial derivatives on Q . Then

$$\int_S \vec{F} \cdot \vec{n} d\sigma = \int_Q \nabla \cdot \vec{F} dV$$

This states that the flux through a closed surface equals the total divergence throughout the solid.

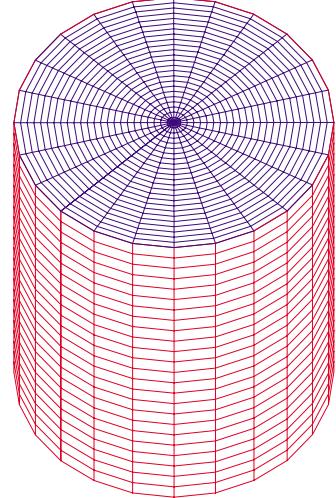
EXAMPLE 1: Consider the solid cylinder

$Q = \{(x, y, z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$ and

the vector field $\vec{F}(x, y, z) = \langle xz, y, z \rangle$.

The divergence of \vec{F} is $\nabla \cdot \vec{F} = z + 1 + 1 = z + 2$.

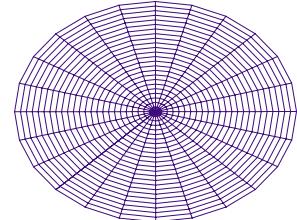
$$\begin{aligned} \int_Q \nabla \cdot \vec{F} dV &= \int_0^{2\pi} \int_0^1 \int_0^1 (z+2)r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \left[\frac{z^2}{2} + 2zr \right]_{z=0}^{z=1} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \frac{5}{2}r dr d\theta \\ &= \int_0^{2\pi} \frac{5}{4}r^2 \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{5}{4} d\theta \\ \int_Q \nabla \cdot \vec{F} dV &= \frac{5\pi}{2} \end{aligned}$$



Now we will compare our answer with the sum of three surface integrals, each with the normal vector \vec{n} pointing outward.

bottom $z = 0, \quad \vec{n} = -\vec{k}, \quad \vec{F}(x, y, 0) = \langle 0, y, 0 \rangle, \quad \vec{F} \cdot \vec{n} = 0$ so

$$\int_{\text{bottom}} \vec{F} \cdot \vec{n} d\sigma = 0$$



top $z = 1, \quad \vec{n} = \vec{k}, \quad \vec{F}(x, y, 1) = \langle x, y, 1 \rangle, \quad \vec{F} \cdot \vec{n} = 1$ so

$$\int_{\text{top}} \vec{F} \cdot \vec{n} d\sigma = \int_{\text{top}} 1 d\sigma = 1(\text{area of top}) = \pi$$

side As we saw in an early example, the side is parametrized by

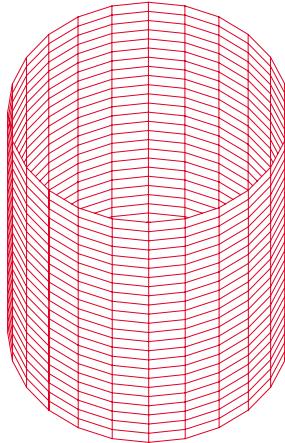
$$g(\theta, z) = \begin{matrix} \cos \theta & 0 \leq \theta \leq 2\pi \\ \sin \theta & 0 \leq z \leq 1 \\ z & \end{matrix} \quad \frac{\partial g}{\partial \theta} = \begin{matrix} -\sin \theta & 0 \\ \cos \theta & 0 \\ 0 & 1 \end{matrix} \quad \frac{\partial g}{\partial z} = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$$

$$\frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial z} = \begin{matrix} -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \vec{i} & \vec{j} & \vec{k} \end{matrix}$$

$$\vec{n} = \langle \cos \theta, \sin \theta, 0 \rangle \quad \text{Is this a surprise?}$$

$$\vec{F} \cdot g(\theta, z) = \langle z \cos \theta, \sin \theta, z \rangle$$

$$\vec{F} \cdot \vec{n} = z \cos^2 \theta + \sin^2 \theta$$



So our surface integral is

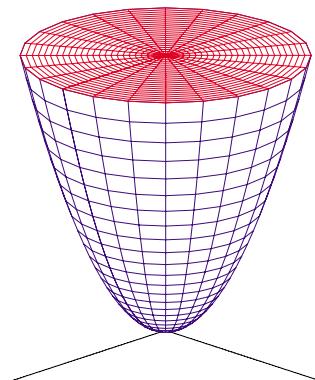
$$\begin{aligned} \int_0^{2\pi} \int_0^1 z \cos^2 \theta + \sin^2 \theta dz d\theta &= \int_0^{2\pi} \int_0^1 \frac{z^2}{2} \cos^2 \theta + z \sin^2 \theta d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \cos^2 \theta + \sin^2 \theta d\theta \\ &= \int_0^{2\pi} \frac{1}{4} (1 + \cos 2\theta + \frac{1}{2}(1 - \cos 2\theta)) d\theta \\ &= \int_0^{2\pi} \frac{3}{4} - \frac{1}{4} \cos 2\theta d\theta = \frac{3}{4}\theta - \frac{1}{8} \sin 2\theta \Big|_0^{2\pi} \\ &= \frac{3\pi}{2} \end{aligned}$$

Thus,

$$\int_S \vec{F} \cdot \vec{n} d\sigma = 0 + \pi + \frac{3\pi}{2} = \frac{5\pi}{2}$$

which agrees with the volume integral, as predicted by the Divergence theorem.

EXAMPLE 2 Let's compare the values of the integrals of the Divergence theorem for the vector field $\vec{F}(x, y, z) = \langle 1, 1, z \rangle$ for the solid Q that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.



Volume integral

$\nabla \cdot \vec{F} = 1$ so

$$\begin{aligned}
\nabla \cdot \vec{F} dV &= \int_Q 1 dV \\
&= \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{z=x^2+y^2}^{z=4} 1 dz dx dy \\
&= \int_0^{2\pi} \int_0^r \int_{r^2}^{r^2} r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^r 4r - r^3 dr d\theta \\
&= \int_0^{2\pi} 4 d\theta
\end{aligned}$$

$$\int_Q \nabla \cdot \vec{F} dV = 8\pi$$

Surface integrals

(a) Top $z = 4$, $\vec{n} = \vec{k} = \langle 0, 0, 1 \rangle$, $\vec{F}(x, y, 4) = \langle 1, 1, 4 \rangle$, $\vec{F} \cdot \vec{n} = 4$

$$\int_S 4 d\sigma = 4(4\pi) = 16\pi$$

(b) Bottom $z = x^2 + y^2$, but we will parametrize in polar coordinates.

$$g(r, \theta) = \begin{cases} r \cos \theta \\ r \sin \theta \\ r^2 \end{cases} \quad \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} = \langle -2r^2 \cos \theta, -2r^2 \sin \theta, r \rangle$$

It is important to note that our normal vector points inward, so we must choose its negative! Thus

$$\vec{n} = \langle 2r^2 \cos \theta, 2r^2 \sin \theta, -r \rangle$$

We have $\vec{F}(g(r, \theta)) = \langle 1, 1, r^2 \rangle$, $\vec{F} \cdot \vec{n} = 2r^2 \cos \theta + 2r^2 \sin \theta - r^3$, so

$$\begin{aligned}
\int_0^{2\pi} \int_0^2 2r^2 \cos \theta + 2r^2 \sin \theta - r^3 dr d\theta &= \int_0^{2\pi} \left[\frac{2r^3}{3} \cos \theta + \frac{2r^3}{3} \sin \theta - \frac{r^4}{4} \right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{16}{3} \cos \theta + \frac{16}{3} \sin \theta - 4 d\theta \\
&= -8\pi
\end{aligned}$$

And as was predicted,

$$\int_S \vec{F} \cdot \vec{n} d\sigma = 16\pi - 8\pi = 8\pi.$$

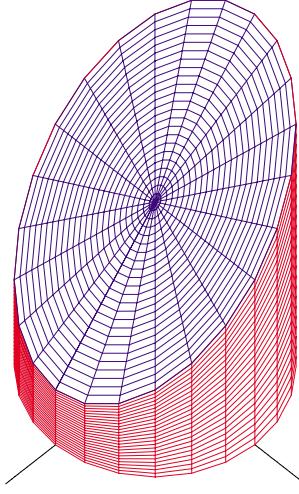
Example 3, Maple: We will illustrate the equivalence of the two integrals in the Divergence Theorem with a challenging problem. The solid is bounded by the oblique plane $2x + z = 6$, the vertical cylinder $x^2 + y^2 = 4$, and the xy -plane. The vector function is $\vec{F}(x, y, z) = \langle x^2 + z, xy + 2z, 3y \rangle$. We will begin by sketching the solid, then compute the divergence over the solid, and end with the three surface integrals. The easiest approach is to use cylindrical coordinates.

```

> restart: with(student): with(plots): with(linalg):
> S:=cylinderplot([2, theta, z], theta=0..2*Pi, z=0..6-2*2*cos(theta), color=red):
> T:=cylinderplot([r, theta, 6-2*r*cos(theta)], r=0..2, theta=0..2*Pi, color=blue):
> xaxis:=spacecurve([t, 0, 0], t=0..3, color=black):
> yaxis:=spacecurve([0, t, 0], t=0..3, color=black):

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```
> display(S, T, xaxis, yaxis);
```



```
> F:=(x,y,z)->[x^2+z, x*y+2*z, 3*y];
       $F := (x, y, z) \rightarrow [x^2 + z, xy + 2z, 3y]$ 
> divF:=divergence(F(x,y,z), [x,y,z]);
       $\text{div}F := 3x$ 
> grand:=subs(x=r*cos(theta), y=r*sin(theta), z=z, divF);
       $\text{grand} := 3r \cos(\theta)$ 
> divint:=TripleIntegrate(grand*r, z=0..6-2*r*cos(theta), r=0..2, theta=0..2*Pi);
       $\text{divint} := \int_0^{2\pi} \int_0^2 \int_0^{6-2r \cos(\theta)} 3r^2 \cos(\theta) dz dr d\theta$ 
> divanswer:=value(divint);
       $\text{divanswer} := -24\pi$ 
```

It should be obvious that we have just finished the easy half of this problem. We will work from top to bottom on the surfaces. We begin with the oblique plane. Note how u plays the role of r and v that of θ .

```
> g1:=vector([u*cos(v), u*sin(v), 6-2*u*cos(v)]);
       $g1 := [u \cos(v), u \sin(v), 6 - 2u \cos(v)]$ 
> g1u:=map(diff, g1, u);
       $g1u := [\cos(v), \sin(v), -2 \cos(v)]$ 
> g1v:=map(diff, g1, v);
       $g1v := [-u \sin(v), u \cos(v), 2u \sin(v)];$ 
> fcp1:=crossprod(g1u, g1v);
       $fcp1 := [2 \sin(v)^2 u + 2 \cos(v)^2 u, 0, \cos(v)^2 u + \sin(v)^2 u]$ 
> fcp1:=simplify(op(fcp1), symbolic);
       $fcp1 := [2u, 0, u]$ 
> Fatg1uv:=F(op(g1));
       $\text{Fatg1uv} := [u^2 \cos(v)^2 + 6 - 2u \cos(v), u^2 \cos(v) \sin(v) + 12 - 4u \cos(v), 3u \sin(v)]$ 
> grand1:=innerprod(Fatg1uv, fcp1);
       $\text{grand1} := 2u^3 \cos(v)^2 + 12u - 4u^2 \cos(v) + 3u^2 \sin(v)$ 
> ans1:=DoubleIntegrate(grand1, u=0..2, v=0..2*Pi);
       $\text{ans1} := \int_0^{2\pi} \int_0^2 2u^3 \cos(v)^2 + 12u - 4u^2 \cos(v) + 3u^2 \sin(v) du dv$ 
> flux1:=value(ans1);
       $\text{flux1} := 56\pi$ 
```

Now we will work on the curved side.

```
> g2:=vector([2*cos(u), 2*sin(u), v]);
       $g2 := [2 \cos(u), 2 \sin(u), v]$ 
```

```

> g2u:=map(diff,g2,u);
g2u := [-2 sin(u), 2 cos(u), 0]
> g2v:=map(difff,g2,v);
g2v := [0, 0, 1]
> fcp2:=crossprod(g2u,g2v);
fcp2 := [2 cos(u), 2 sin(u), 0]
> Fatg2uv:=F(op(g2));
Fatg2uv := [4 cos(u)^2 + v, 4 cos(u) sin(u) + 2v, 6 sin(u)]
> grand2:=innerprod(Fatg2uv,fcp2);
grand2 := 8 cos(u)^3 + 2 cos(u)v + 8 cos(u) sin(u)^2 + 4 sin(u)v
> grand2:=simplify(grand2,symbolic);
grand2 := 2 cos(u)v + 8 cos(u) + 4 sin(u)v
> ans2:=Doubleint(grand2,v=0..6-2*2*cos(u),u=0..2*Pi);
ans2 := 
$$\int_0^{2\pi} \int_0^{6-4\cos(u)} 2 \cos(u)v + 8 \cos(u) + 4 \sin(u)v \, dv \, du$$

> flux2:=value(ans2);
flux2 := -80\pi

```

Let's think about the bottom surface for a moment. The *outward* unit normal vector is $-\vec{k} = \langle 0, 0, -1 \rangle$.

```

> Fonbot:=F(x,y,0);
Fonbot := [x^2, xy, 3y]
> n:=vector([0,0,-1]);
n := [0, 0, -1]
> grand3:=innerprod(Fonbot,n)
grand3 := -3y
> grand3:=subs(x=r*cos(theta),y=r*sin(theta),grand3)
grand3 := -3r sin(\theta)
> ans3:=Doubleint(grand3*r,r=0..2,theta=0..2*Pi);
ans3 := 
$$\int_0^{2\pi} \int_0^2 -3r^2 \sin(\theta) \, dr \, d\theta$$

> flux3:=value(ans3);
flux3 := 0
> Fluxtotal := flux1+flux2+flux3;
Fluxtotal := -24\pi

```

Once again, the divergence integral is seen to be the same as the flux integral. At this point, the reader should begin to develop an appreciation for how the divergence integral is usually easier to evaluate than the surface integral(s).

C3M17 problems

1. Use Maple to evaluate the flux integral, $\int_S \vec{F} \cdot \vec{n} \, d\sigma = \int_D \vec{F}(g(u,v)) \cdot \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \, du \, dv$ directly. $\vec{F}(x,y,z) = \langle y, -x, 1 \rangle$, \vec{n} outward, S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Use Maple and the Divergence Theorem to evaluate the given flux integrals, $\int_S \vec{F} \cdot \vec{n} \, d\sigma$. (Do NOT evaluate the integrals directly!)

2. $\vec{F}(x,y,z) = \langle 3x^2, xy, z \rangle$, S bounds the solid $Q = \{(x,y,z) : x + y + z \leq 1, x, y, z \geq 0\}$
3. $\vec{F}(x,y,z) = \langle x^3, y^3, z^3 \rangle$, Q is the solid above the cone $z^2 = x^2 + y^2$ and below the sphere $x^2 + y^2 + z^2 = 9$. Hint: use spherical coordinates.

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