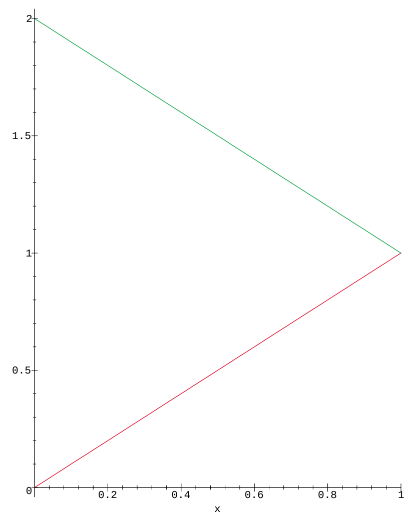
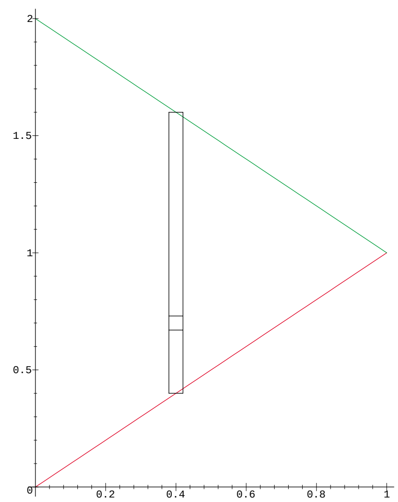
C3M8

Setting up Double Integrals

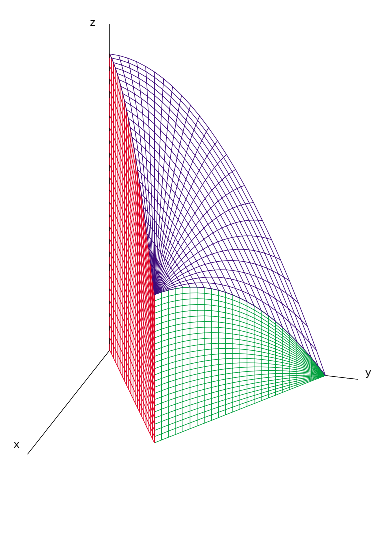
We will begin by drawing a region and showing how the geometry of the region leads to the set-up of an iterated integral. The operative phrase is "curve-to-curve, point-to-point". As we promised in an earlier session, we will begin with a region that is bounded by $y = x,\ y = 2 - x,\ x = 0$. The Maple syntax shown will produce the plot on the left. The right-hand plot is provided to help explain why the region requires that $dA = dy\,dx$.

```
> restart:    with(plots):
> plot([x,2-x],x=0..1);
```

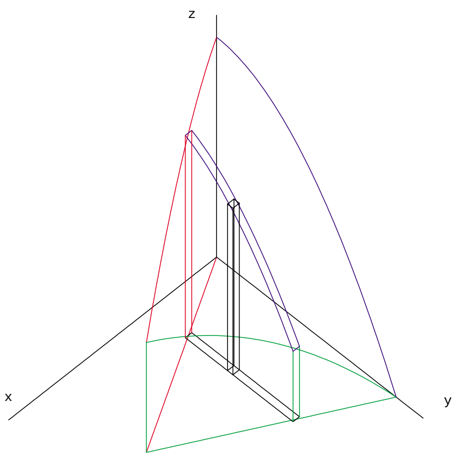


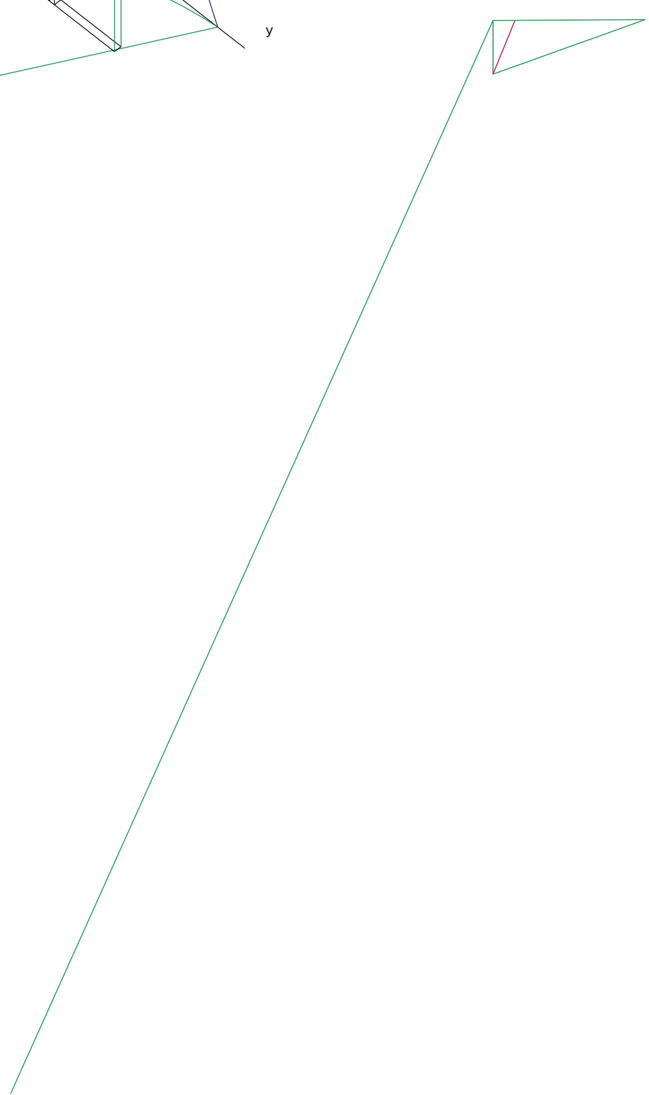


If you top this region with $z = 4 - x^2 - y^2$, then you get the figure below.



If we draw horizontal lines across the region we see that the lines will always start at $x = 0$, which is the y-axis, but they can end on two different curves, $y = x$ and $y = 2 - x$. This would mean that integrating first with respect to x, which changes from left to right, would violate the 'curve-to-curve' concept for our inside integral. As you see in the right-hand plot, every vertical line begins at $y = x$ and ends on $y = 2 - x$. Think of y, which increases by moving up, and you see that the inside integral can be with respect to y. Visualize the small rectangle as it slides up the longer one. Now, imagine that little rectangle as the base of a vertical column inside the solid. As the volume of each column is added up along the long rectangle we obtain the volume of a "slab" that is parallel to the yz-plane. Think of the first integration as determining the volumes of all those slabs. The second integration adds up the volumes of the slabs to get the volume of the solid. At what number does the 'first' slab occur? At what number does the 'last' slab occur. That is the "point-to-point" part of the process. This idea is illustrated in the plots that follow. First, a value is established for each column along the strip that is parallel to the y-axis. The diagram on the left shows the column in outline form and the one on the right shows the column as a solid. It is important here to realize that the values for each column are summed between two curves, $y = g(x)$ and $y = h(x)$, as the first step in the process.





must complete this iterated integral in the format shown:

$$\int_{x=}^{x=}\int_{y=}^{y=} 4 - x^2 - y^2\, dy\, dx = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} 4 - x^2 - y^2\, dy\, dx$$

This means that the inner integral will be

$$\int_{y=x}^{y=2-x} 4 - x^2 - y^2\, dy$$

Think of a vertical line sliding from left to right over the region. The first x value that occurs is $x = 0$ and the last is $x = 1$. So our double integral becomes

$$\int_{x=0}^{x=1}\int_{y=x}^{y=2-x} 4 - x^2 - y^2\, dy\, dx$$

BEWARE! In Maple, we must be sure to put the inner integral limits first and the outer ones last.

```
> A:=Doubleint(4-x^2-y^2,y=x..(2-x),x=0..1);
```

$$A := \int_{0}^{1}\int_{x}^{2-x} 4 - x^2 - y^2\, dy\, dx$$

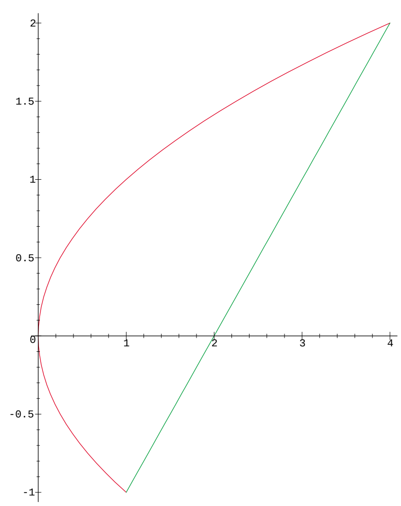
Incidentally, the value of this integral is $\frac{8}{3}$. Compare the syntax for the double integral to the equivalent non-parametric syntax we would have used to plot the top in the graphing section. The similarities are obvious.

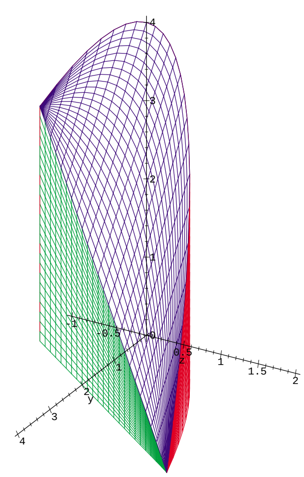
```
> A3:=plot3d(4-x^2-y^2,y=x..(2-x),x=0..1,color=blue):
```

Example: Sketch the solid bounded by $x = y^2$, $x = y + 2$, $z = 0$, $x + z = 4$ and find its volume.

Eliminating x from the first two equations yields $y^2 = y + 2$, $0 = y^2 - y - 2 = (y - 2)(y + 1)$, so the vertical parabolic surface meets the vertical plane at $y = -1$ and $y = 2$. The horizontal plane $z = 0$ will serve as a bottom and the oblique plane $x + z = 4$ as a top. The domain in the plane is easy to plot parametrically. The output is on the left below.

```
> A1:=plot([y^2,y,y=-1..2],color=red):
> A2:=plot([y+2,y,y=-1..2],color=green):
> display(A1,A2);
```





We could set up our iterated integral now, but let's plot the solid first. We can use $A1$ and $A2$ to help us. The figure on the right above was produced by:

```
> A3:=plot3d([y^2,y,z],y=-1..2,z=0..4-y^2,color=red):
> A4:=plot3d([y+2,y,z],y=-1..2,z=0..(2-y),color=green):
> A5:=plot3d([x,y,4-x],x=y^2..(y+2),y=-1..2):
> display(A3,A4,A5);
```

Consider horizontal and vertical lines and how they would meet the region. Vertical lines could go from one side of the parabola to the other or from the line to the parabola, suggesting that one should not integrate with respect to y first. Horizontal lines would go from the parabola to the line. Ahah! We can integrate with respect to x first. The integrand is found by solving $x + z = 4$ for z, and that means that $z = 4 - x$. We show the three stages of setting up the integral.

□