C3NOTES

A Calculus and MAPLE 7 Supplement

SM221 **Spring, 2002**

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C3NOTES

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Calculus III **C3Notes** **Spring, 2001-2002**

by Professor C. E. Moore

Introduction to Maple 7

Open Maple 7 and obtain a blank worksheet. We are going to begin by establishing a format for each Maple assignment that is to be submitted in Calculus III. For those who took Calculus II in the fall this section should look familiar. Do not type the “<” or “>” shown to identify your entries. And <Enter> means the “Enter” key. As you begin, the worksheet is in “math mode”, so ‘click’ on the $\boxed{\mathbf{T}}$ to switch into “text mode”. Type <C2M1> <Enter> and then highlight C3M1 and click on the middle of the three boxes to the right of $\boxed{\mathbf{B}}$ $\boxed{I}$ $\boxed{\underline{u}}$ so as to center C3M1. The left of these three buttons left-justifies text and the right one right-justifies it. Now,
<down arrow>, then type your name and section as shown.
<Midn Your Name> <Enter>
<Section> <Your section> <Enter>
Having completed this, highlight the three lines and then click on $\boxed{\mathbf{B}}$ to boldface everything. This is the format you should use for all Maple assignments to be handed in. For example, you should see

C3M1

Midn John Doe
Section 1234

You may eliminate the brackets on the left by the function key <F9>. To return to math mode, click on the $\boxed{[>}$. If we wanted to type a math formula while in text mode we would click on $\boxed{\Sigma}$. Later in this section we will discuss *palettes* which allow you to select commands from a menu and avoid using Maple syntax. It is the contention of the author of these notes that learning some Maple syntax is beneficial to the student, so even though you may accomplish the same things by clicking on a symbol, we will show you the syntax that would otherwise be hidden.

For efficiency, Maple groups sets of commands into packages which must be listed when needed. For many calculus operations we will need “student”. In math mode, lines in Maple end with a semicolon or colon. If you put a colon after the line then the display of the output is suppressed. Please type the command lines below in a new worksheet exactly as you see them and note the output. This work is for your benefit and is not intended to be handed in.

```
> with(student):
> A:=Int(x^2,x);
> value(A);
> B:=int(x^2,x);
> C=Int(x^2,x=2..5);
> value(C);
```

In the first line, the name “A” is being assigned to the unevaluated integral, $\int x^2\,dx$. Then we find the value of A. By using the lowercase “i” we assign the name “B” to the value of the integral, rather than the inert expression that is not yet evaluated. By eliminating the colon, we have established an equation rather than assigning a name “C” to the integral. Think of it this way. The integral is being assigned to a piece of memory named “A”, another piece of memory is assigned the value of that integral and is named “B”, while an equation involving C and an unevaluated definite integral is established. Evaluating “C” is meaningless because there is no piece of memory with that name. Go back, insert the colon as was done with A and hit <Enter> and then another <Enter>. The exact value of the integral, 7, should appear. Be very careful of these subleties, because Maple cannot read your mind. What happens when you put a semicolon after ‘`with(student)`’ instead of the colon?

In that same worksheet enter these command lines and observe the output.

```
> value(Pi^2/6);
> evalf(Pi^2/6);
> S100:=Sum(1/k^2,k=1..100);
> value(S100);
> evalf(S100);
```

The `evalf` command converts an exact numerical expression to a floating point number.

It is very important to know how to define a function, how to define an expression, and the difference between them. To define the function $f(x) = 4 - 2x - x^2$ please enter:

```
> f:=x->4-2*x-x^2;
```

And then evaluate f at $x = 4/3$ by:

```
> f(4/3);
```

Now define an expression *expr* by:

```
> expr:=4-2*x-x^2;
```

In order to evaluate an expression at a value, we must substitute into the expression. Try:

```
> subs(x=4/3,expr);
```

And when we try to use an expression as if it were a function, we get garbage. Try:

```
> garbage:=expr(4/3);
```

It will be very useful later to be able to make a function out of an expression. The syntax for this is puzzling. Entering these commands should produce these results:

```
> P:=x^2+cos(x);
```

$$P := x^2 + \cos(x)$$

```
> G:=unapply(P,x);
```

$$G := x \to x^2 + \cos(x)$$

```
> G(Pi);
```

$$\pi^2 - 1$$

We see that G is a function and that P is the image of x.

There are two operations that are very basic in calculus - differentiation and integration, or anti-differentiation. Expressions in x and t can be differentiated with respect to either variable so we must remember to specify the variable. Using the expression P from above we have

```
> Pprime:=diff(P,x);
```

$$Pprime := 2x - \sin(x)$$

And if we integrate P

```
> Pint:=int(P,x);
```

$$Pint := \frac{x^3}{3} + \sin(x)$$

Maple graphics are versatile and easy to use. We can get a quick plot of f on $[-2, 3]$ by:

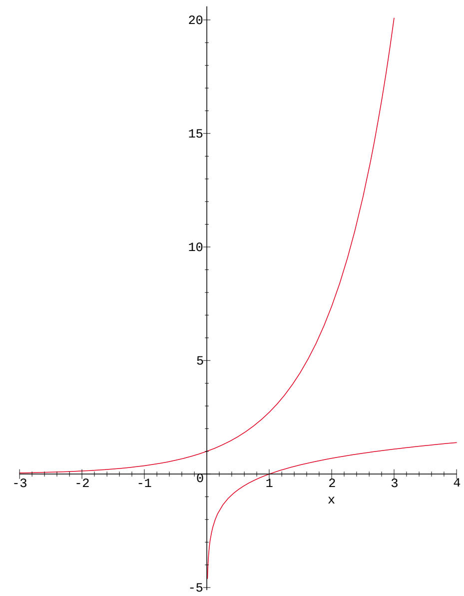
```
> plot(f(x),x=-2..3,color=blue);
```

Note how we used $f(x)$, not just f in this plot. To plot the function G from above we could use P, which is an expression, or $G(x)$ and obtain the same result.

```
> plot(P,x=0..Pi,color=magenta);
```

Multiple plots. It will be extremely useful later to have the ability to plot two functions with different domains. We need the graphics package "plots". Note the use of colons when we give a plot a name. We do not want Maple to give output at that time. The exponential function, $y = e^x$, is accessed in Maple by `exp(x)`. You are encouraged to enter the commands below in a Maple worksheet exactly as shown.

```
> with(student):    with(plots):
> A:=plot(exp(x),x=-3..3):
> B:=plot(ln(x),x=.01..4):
> display(A,B);
```



Now let's do some of the same steps by using a palette. There are three palettes and you begin by clicking on "View", then "Palettes". If you need symbols, select that palette, but now we choose "expressions". On your command line type

`A:=`

and then click on the box with the integral symbol $\boxed{\int a}$. Then click on the box with $\boxed{a^b}$. On your command line the cursor appears where you want x inserted, so you type $< x >$, and then **move to the next entry position by using the 'Tab' key**. Enter $< 2 >$, <Tab>, and then the variable of integration, $< x >$ and <Enter>. Continue by working through the same steps as above by using palettes.

Suppose you have a question about some aspect of using Maple and the syntax is confusing you. On the command line simply type

`<?topic>`<Enter>

and Maple will display what you need to learn about "topic". You can copy from the help page and paste on the command line as needed.

C3M1

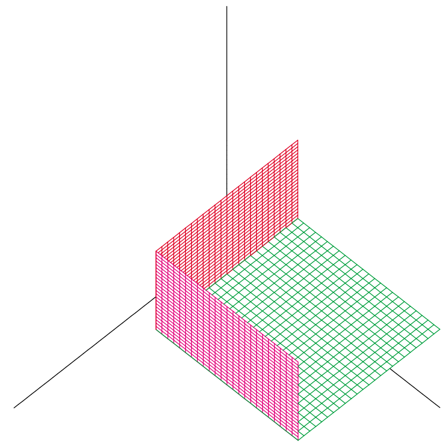
Three-Dimensional Graphics

There are several packages of commands in Maple that we will use frequently. For many calculus operations we will need "student". For vector operations the package is "linalg". One of the real strengths of Maple is its ability to graph curves and surfaces in a three-dimensional coordinate system. We need the package "plots" in order to do this. There are two basic ways to use the command "plot3d". The first plots $z = f(x, y)$, while the second and most versatile plots $[x, y, z]$ parametrically with x, y, and z as functions of two variables. The reader is encouraged to input the commands being discussed and to try the suggestions to see the effects that they have. It may be to your advantage to save the Maple work that you type in to test because you may be able to cut, paste, and edit them when you need similar entries later. Remember, after editing a worksheet you should execute all the commands from top to bottom. It is easiest to go to "EDIT", "EXECUTE", and then choose "WORKSHEET". Or, you may start at the top of a worksheet and hit <Enter> all the way down. The command `restart:` clears the Maple kernel of all internal memory. Some put `restart:` on the first line of a worksheet before any packages such as 'student' or 'plots' so that confusion is avoided if one <Enter>'s from the first line all the way through. Do NOT put `restart:` on a line AFTER you have listed packages because that will erase the packages that you think have been included.

Example 1: Draw three faces of the rectangular box defined by
$[0, 2] \times [1, 3] \times [1, 2] = \{(x, y, z) : \ 0 \le x \le 2, \ 1 \le y \le 3, \ 1 \le z \le 2\}$ and include coordinate axes.

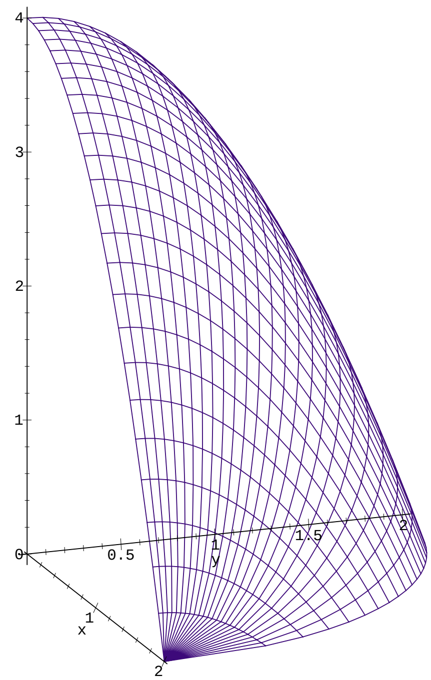
Let's start with the x-axis. The command 'spacecurve' has the parametric form of a curve as its argument, along with the range and choice of color. All we need here is $[t, 0, 0]$ to generate a portion of the axis. We must suppress the output with a ':' so that all the pieces may be displayed at one time. In the face labelled 'A' the y values is 1, and x and z may vary. Analyze the lines below and predict the output of each.

```
> with(plots):
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> zaxis:=spacecurve([0,0,t],t=0..3,color=black):
> A:=plot3d([x,1,z],x=0..2,z=1..2,color=red):
> B:=plot3d([x,y,1],x=0..2,y=1..3,color=green):
> C:=plot3d([2,y,z],y=1..3,z=1..2,color=magenta):
> display(xaxis,yaxis,zaxis,A,B,C);
```



Example 2: Draw the portion of the paraboloid $z = 4 - x^2 - y^2$ that is over the quarter-disk of radius 2 in the first quadrant.

```
> with(plots):
> plot3d(4-x^2-y^2,x=0..2,y=0..sqrt(4-x^2),color=blue);
```



Note how we kept x between 0 and 2, but y was between 0 and $\sqrt{4 - x^2}$. At this point you should have a three-cornered sheet of blue lines appear. Move the pointer onto the figure and click once. A rectangle should appear around the figure and a new set of menu options are seen above. The button **1:1** adjusts the ratios of the axes. There are four red spheres next to **1:1**, click on each of them and note how you get different ways of showing the axes on the figure, with one option being no axes. There are 7 black spheres to the left of the four red ones. One at a time, click on each of the spheres. You may wish to end this sequence with the middle sphere. Now, click on the figure and hold down the left button of the mouse. Move the mouse so as to move the pointer and note how the figure rotates. On the left end of the line above with the spheres you will find two angles displayed. As you rotate the figure the values of those angles changes and are displayed accordingly. The angle on the left is Θ, or in lower case θ, and measures rotation about the z-axis. When $\Theta = 0$ you are looking down the x-axis. The second angle is Φ, or in lower case ϕ or φ, and measures how much the z-axis has deflected.

When $\Phi = 0$ you are looking down the z-axis from above. When $\Theta = 45$ and $\Phi = 75$, you have the x-axis to your left and the y-axis to your right equally, while the z-axis is tipped forward so as to give you the usual perspective one gets when sketching in 3-d. This will all make more sense to you after you have been introduced to cylindrical and spherical coordinate systems.

Before you move on, click carefully on the lower (right-hand) corner of the rectangle and drag it towards the opposite corner until you have a small square of about two inches, release and the figure will redraw within the box. You are expected to reduce the size of your plots in the assignments so as to save paper.

We would have gotten the same result from

```
> plot3d([x,y,4-x^2-y^2],x=0..2,y=0..sqrt(4-x^2),color=blue);
```

which is the parametric form of the same surface. The value of the parametric form is that vertical surfaces are easily handled, but of course they cannot occur as $z = f(x, y)$. Our focus here is on the quadric surfaces such as paraboloids, hyperboloids, ellipsoids, spheres and cones. But in order to display them it is best to learn how to show the result of cutting these ‘solids’ with planes. When one variable is held constant we have simply intersected the figure with a plane that is parallel to the coordinate plane of the remaining two variables. The result is called a ‘trace’.

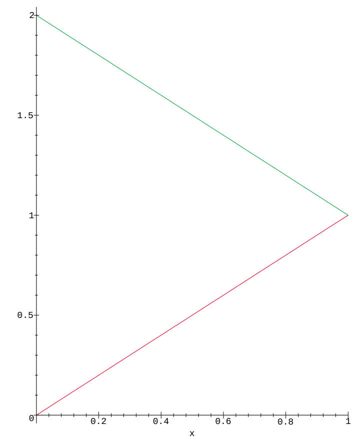
Example 3: Draw the solid figure bounded on the sides by $y = x$, $y = 2 - x$, and $x = 0$, below by $z = 0$, and above by $z = 4 - x^2 - y^2$.

We will use this same example to introduce double integrals later, so a little effort here will be helpful. If you draw the lines $y = x$ and $y = 2 - x$ in the first quadrant and then draw the line $x = 0$, you will see that a triangle has been formed.

BEWARE! The line $x = 0$ is vertical and is **NOT** the x-axis. Note that using the x-axis as a boundary defines a different triangle. This error frequently occurs.

Please observe the result of the following plot on your screen.

```
> plot([x,2-x],x=0..1);
```



The top line should be green and the bottom should be red. If the triangle was blue then you would have a good view looking down on the solid. But the red and green lines will be edges of the vertical surfaces. The top of our solid is the paraboloid $z = 4 - x^2 - y^2$. Keep this in mind when you consider the upper bounds for z when plotting the sides. How do we know when a ‘side’ is vertical? The variable z will be missing from that equation. We begin with a surface in the vertical plane $y = x$ so that every y is replaced by x.

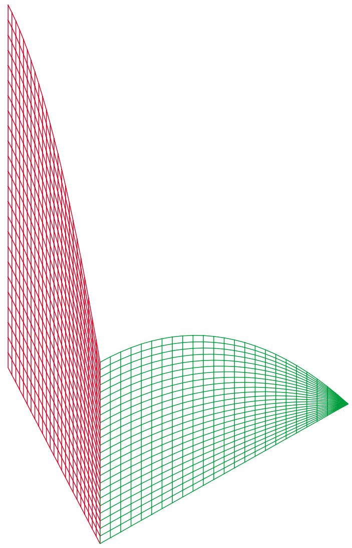
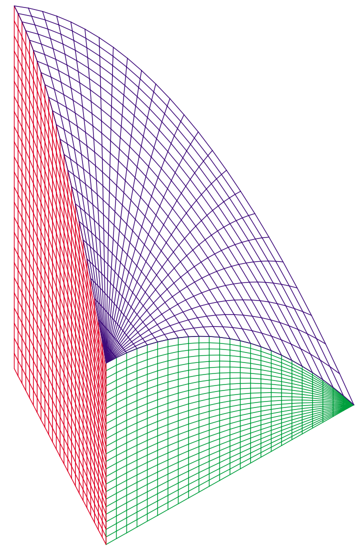
```
> plot3d([x,x,z],x=0..1,z=0..4-2*x^2, color=red);
```

You are probably a little puzzled by the result - a vertical red line. Using what you learned up above, rotate the figure to the right 30 degrees or so. Now you should see a surface. Maple orients the initial plot so that the vertical plane $y = x$ is directly towards the viewer, i.e. $\Theta = 45$. If you examine our command you will see `[x,x,z]` which means our plot lies in the plane $y = x$ since the first and second coordinates are the same. We wish to add another plane to the situation and we will draw it separately before combining our plots.

```
> plot3d([x,2-x,z],x=0..1,z=0..4-x^2-(2-x)^2,color=green);
```

Because this surface resides in the plane $y = 2 - x$, wherever y would occur we have replaced it by $2 - x$. In particular, note the upper bound for z. To display these plots jointly, we must give the plots names and suppress their outputs with colons at the ends of their command lines.

```
> A1:=plot3d([x,x,z],x=0..1,z=0..4-2*x^2, color=red):
> A2:=plot3d([x,2-x,z],x=0..1,z=0..4-x^2-(2-x)^2,color=green):
> display(A1,A2);
```

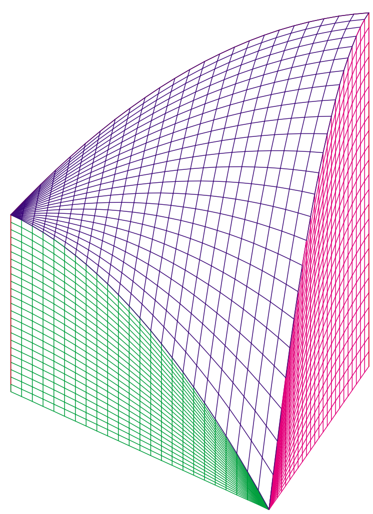
 

After rotating and including the axes you should see the figure on the left above. We add our top, suppressing its output, and display all three together. When drawing the top, observe that for any fixed x, y will vary between the values x and $2-x$, i.e. "curve-to-curve". The output is above on the right.

```
> A3:=plot3d([x,y,4-x^2-y^2],x=0..1,y=x..2-x,color=blue):
> display(A1,A2,A3);
```

We see two sides and a top of the paraboloid. The third vertical side, if it were needed, would result from the plot labelled "A4". This side is viewed by rotating to the left.

```
> A4:=plot3d([0,y,z],y=0..2,z=0..4-y^2,color=magenta):
> display(A1,A2,A3,A4);
```



Example 4: Display the portion of the hyperboloid of one sheet $x^2+y^2-z^2=1$ for which $x \le 0$, $y \ge 0$ and $-1 \le z \le 1$.

Looking at this solid from a point out on the x-axis we would see a flat surface bounded on the left by a vertical line, (the z-axis), on the top and bottom by horizontal lines (edges of planes) $z=-1$, $z=1$, and on the right by a hyperbola that bends to the left in the middle. We will draw this surface third. Rotate this surface about the z-axis 90° and you have a surface that will be hidden from our view, but it serves as the domain of our parametrization of the curved surface. We realize that we cannot draw the curved surface as a function of x and y. So, let's solve for y in terms of x and z. We are being careful to go 'curve-to-curve and point-to-point' here so x must vary from 0 to x as a function of z.

$$x^2 + y^2 - z^2 = 1 \quad \Rightarrow \quad y^2 = 1 + z^2 - x^2 \quad \Rightarrow \quad y = \sqrt{1 + z^2 - x^2}$$

Let $y = 0$ and solve for x in terms of z : $x^2 = 1 + z^2 \quad \Rightarrow \quad x = -\sqrt{1+z^2}$. In Maple, we have

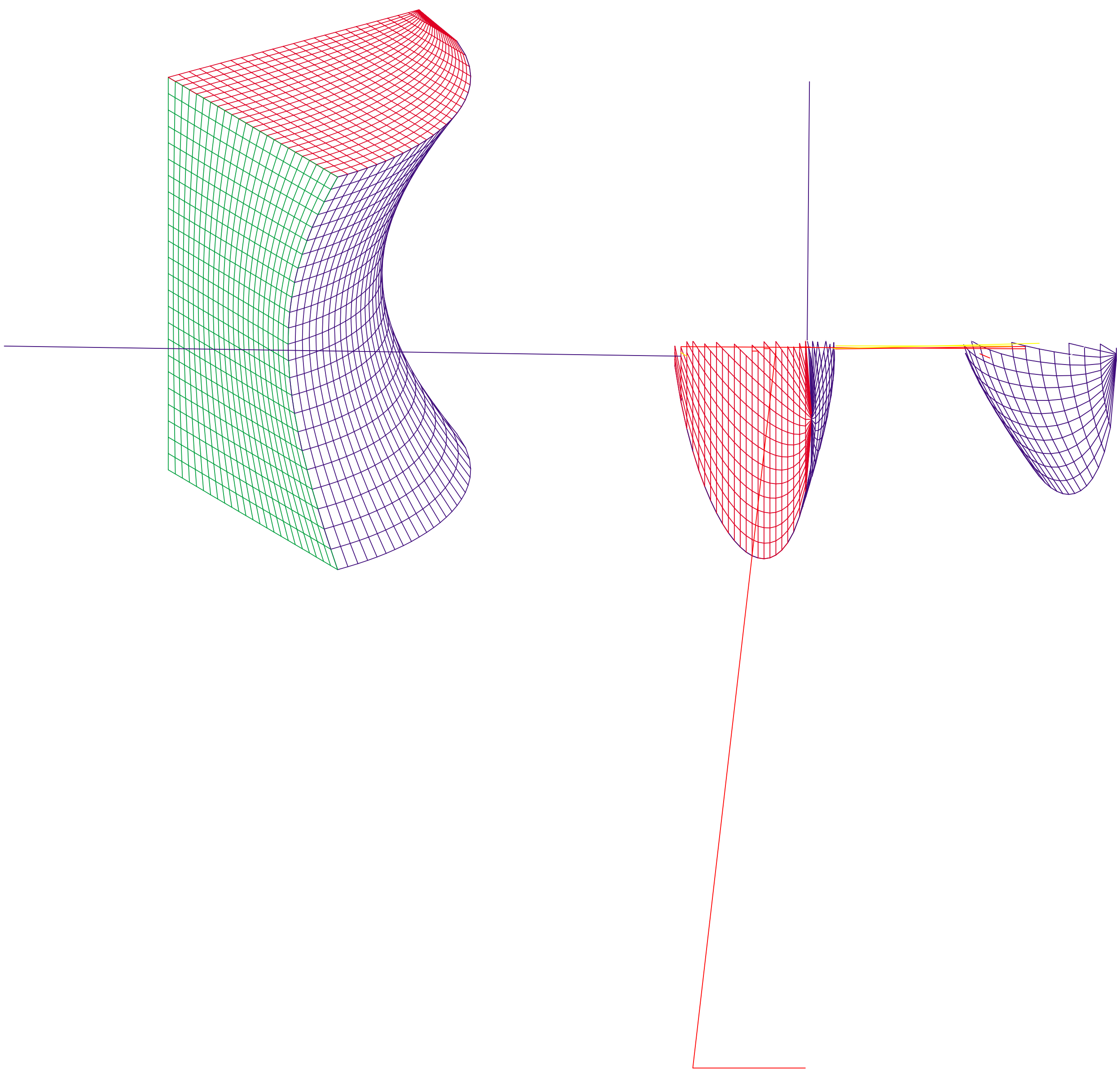
```
> H1:=plot3d([x,sqrt(1+z^2-x^2),z],x=-sqrt(1+z^2)..0,z=-1..1,color=blue):
```

The top is a quarter of a disk. The radius is determined by putting $z = \pm 1$ in the equation of the hyperboloid. We get $x^2 + y^2 - 1 = 1 \quad \Rightarrow \quad x^2 + y^2 = 2$ and

```
> H2:=plot3d([x,y,1],x=-sqrt(2)..0,y=0..sqrt(2-x^2),color=red):
```

The remaining surface we need lives in the plane $x = 0$. As in the domain of `H1`, z cannot be the dependent variable. The display is shown on the left below.

```
> H3:=plot3d([0,y,z], y=0..sqrt(1+z^2),z=-1..1,color=green):
> display(H1,H2,H3);
```



2. R is the region in the first octant between $y = x$ and $y = 2x$ for $x \leq 2$. Also, $0 \leq z \leq 3$.

3. S is the region inside $z = x^2 + y^2$ that is below $z = 5$.

4. T is the region inside the cylinder $x^2 + y^2 = 4$ that is above $z = 0$ and below $z = 9 - x^2 - y^2$.

C3M2

Plotting in Cylindrical and Spherical Coordinates

There are two basic 3d-plotters which will be introduced in this section. The first is '`cylinderplot`' and the second is '`sphereplot`' which use cylindrical coordinates and spherical coordinates as their bases respectively. The syntax for each may involve a standard form or a parametric form. While we may use each form, the parametric is the most useful, especially when the surface to be plotted is vertical. For now we will concentrate on `cylinderplot`. When the surface is defined by $z = f(r, \theta)$, for r between $g(\theta)$ and $h(\theta)$, and θ between a and b we would use the parametric form

```
> cylinderplot([r,t,f(r,t)],r=g(t)..h(t),t=a..b,color=green);
```

It is easy to see that we have used t instead of θ here.

Before we begin it is helpful to understand what results when one of the variables is held constant in cylindrical coordinates. With coordinates $[r, \theta, z]$, a vertical cylinder is produced when r is held constant and the other variables have constant bounds. Then, when θ is constant a rectangular portion of a vertical plane containing the z-axis is determined. Last, if z is constant, a portion of an annular ring that is horizontal results. Think of a piece of pie with a bite taken out of the part closest to the center.

Example 1: Use cylindrical coordinates to sketch the quarter of the vertical cylinder $x^2 + y^2 \leq 1$ bounded above by the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant. Assume that the package 'plots' has been included above in the worksheet.

The top is given by:

```
> roof:=cylinderplot([r,t,sqrt(4-r^2)],r=0..1,t=0..Pi/2,color=blue):
```

The bottom comes from:

```
> floor1:=cylinderplot([r,t,0],r=0..1,t=0..Pi/2,color=green):
```

The curved side of the cylinder has its radius held constant at 1, while the angle and height change.

```
> cwall:=cylinderplot([1,t,z],t=0..Pi/2,z=0..sqrt(3),color=red):
```

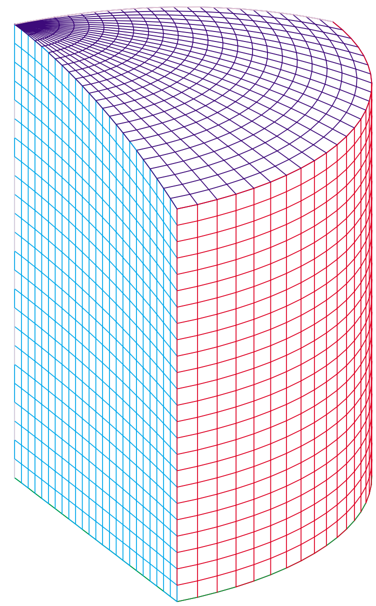
The side that lies in the plane $x = 0$ corresponds to having $\theta = \pi/2$, so:

```
> side1:=cylinderplot([r,Pi/2,z],z=0..sqrt(4-r^2),r=0..1,color=plum):
```

Note how z varied between two curves, $z = 0$ and $z = 4 - r^2$ here and when $y = 0$. The other vertical wall occurs when $\theta = 0$ and:

```
> side2:=cylinderplot([r,0,z],z=0..sqrt(4-r^2),r=0..1,color=cyan):
```

Putting them all together,



```
> display(floor1,roof,cwall,side1,side2);
```

The reader is encouraged to enter the commands above and observe the results. Rotate the solid so that you can see each of the surfaces. Save this example because we will use it later and will observe how to find its volume using a triple integral.

Example 2: Use cylindrical coordinates to plot the solid that lies inside the sphere $x^2+y^2+z^2=9$, above the xy-plane, below the upper portion of the cone $x^2+y^2=z^2$, and excludes the first octant.

We begin by noting that in cylindrical coordinates the sphere is $r^2+z^2=9$ and the cone is $r=z$. By substitution, we see that these surfaces intersect when $r=z=3/\sqrt{2}$. We begin with the cone:

```
> cone2:=cylinderplot([r,t,r],r=0..3/sqrt(2), t=Pi/2..2*Pi,color=blue):
```

Now for the spherical surface:

```
> sph:=cylinderplot([r,t,sqrt(9-r^2)],r=3/sqrt(2)..3,t=Pi/2..2*Pi,color=red):
```

Pay close attention to how we plot the sides. If we tried to let z vary between two curves, what would happen? The angle θ is a constant. We must let r vary between two curves. In a parametric plot of a surface it normally happens that one variable varies between two curves, while the other lies between two values. Begin to think: “Curve-to-curve, point-to-point”.

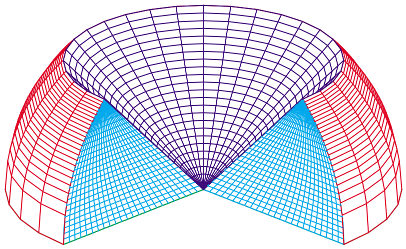
```
> side1:=cylinderplot([r,0,z],r=z..sqrt(9-z^2),z=0..3/sqrt(2),color=cyan):
```

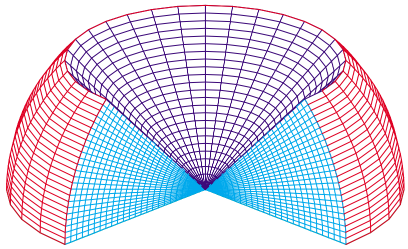
On the other side $\theta=\pi/2$:

```
> side2:=cylinderplot([r,Pi/2,z],r=z..sqrt(9-z^2),z=0..3/sqrt(2),color=cyan):
```

The bottom remains and then we display them all together, with the resulting figure on the left.

```
> floor2:=cylinderplot([r,t,0],r=0..3, t=Pi/2..2*Pi,color=green):
> display(floor2,cone2,sph,side1,side2);
```





Now we turn our attention to spherical coordinates where a point is identified by $[\rho, \theta, \phi]$. The command ‘ `sphereplot(f(theta,phi),theta=g(phi)..h(phi),phi=a..b);`’ is used when $\rho = f(\theta, \phi)$. It is a little easier to use `sphereplot` in the parametric form where one of the variables is a function of the other two. For example, to obtain all of a magenta sphere of radius 2 centered at the origin,

```
> sphereplot([2,theta,phi],theta=0..2*Pi,phi=0..Pi,color=magenta);
```

The value of ρ is the distance to the point from the origin. The angle θ is the same as that used in cylindrical coordinates. Remember, the angle ϕ is measured down from the z-axis to the point. We will begin with the example that we just completed, but will use spherical coordinates. All the variables will lie between constants. Please think about this last statement.

First, let's consider the consequences of holding each of the variables constant in spherical coordinates. In a system with $[\rho, \theta, \phi]$, a portion of a sphere of fixed radius ρ occurs when ρ is a constant. Second, a part of a vertical plane containing the usual z-axis is determined. It would look like a piece of pie on its side. Last, ϕ constant determines a cone with the center axis along the usual z-axis. If $0 < \phi \leq \pi/2$, the cone opens upward and will hold water, while if $\pi/2 \leq \phi < \pi$, it opens downward, like a traffic cone/pylon.

Example 3: Use spherical coordinates to plot the solid that lies inside the sphere $x^2+y^2+z^2 = 9$, above the xy-plane, below the upper portion of the cone $x^2+y^2 = z^2$, and excludes the first octant.

The reader is encouraged to analyze each line and compare carefully with the results of Example 2. This figure is on the right above.

```
> with(plots):
> sph3:=sphereplot([3,theta,phi],theta=Pi/2..2*Pi,phi=Pi/4..Pi/2,color=red):
> cone3:=sphereplot([rho,theta,Pi/4], theta=Pi/2..2*Pi,rho=0..3,color=blue):
> side3:=sphereplot([rho,Pi/2,phi],rho=0..3,phi=Pi/4..Pi/2,color=cyan):
> side4:=sphereplot([rho,0,phi],rho=0..3,phi=Pi/4..Pi/2,color=cyan):
> bottom3:=sphereplot([rho,theta,Pi/2], theta=Pi/2..2*Pi,rho=0..3,color=green):
> display(sph3,cone3,side3,side4,bottom3);
```

The last example is a little more challenging. Recall that in polar coordinates we learned that a vertical line, $x = a$, has the polar equation $r = a\sec\theta$. A horizontal line $y = b$ translates to $r = b\csc\theta$. You are requested to insert the lines below in a worksheet and determine their consequences. In each case ask yourself where $\theta = 0$ and $\theta = \pi/2$ are located in the polar system and where $\phi = 0$ and $\phi = \pi/2$ are located in the spherical system.

```
> sphereplot([2*sec(phi),theta,phi],theta=0..2*Pi,phi=0..Pi/4,color=cyan);
> sphereplot([2*csc(phi),theta,phi],theta=0..2*Pi,phi=Pi/4..3*Pi/4,color=red);
```

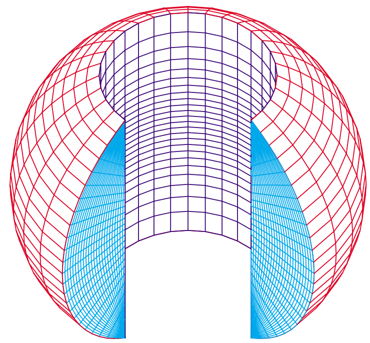
WARNING! If you click on the secant plot and click on an axis sphere, then the system may hang up!

Example 4 Plot the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and is **outside** the cylinder $x^2 + y^2 = 1$.

A three-quarter view of this solid results from:

```
> sph4:=sphereplot([2,theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6):
> cyl4:=sphereplot([csc(phi),theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6):
> side5:=sphereplot([rho,0,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6):
```

```
> side6:=sphereplot([rho,Pi/2,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6):
> display(sph4,cyl4,side5,side6);
```



C3M2 Problems Use cylinderplot and/or sphereplot to plot the solids listed:

1. Q is bounded above by $z = 4$, below by $z = x^2 + y^2$, and on the side by $x^2 + y^2 = 1$.
2. R is a "slice of cheese". It is bounded above by $z = 2$, below by $z = 0$, on the sides by $x = 0$ and $y = x$, and on the outside by $x^2 + y^2 = 4$.
3. S is inside the the sphere $x^2 + y^2 + z^2 = 9$ and above the cone $3z^2 = x^2 + y^2$.
4. T is the portion of the upper hemisphere $x^2 + y^2 + z^2 = 4$ with $z \geq 0$ that lies between the planes $y = x$ and $y = \sqrt{3}x$ in the first octant.

C3M3

Vector Functions and Projectile Motion

A vector-valued function, $\vec{p}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, whose domain is an interval, may be regarded as a position function which depends upon time. So the derivative $\vec{p}\,'(t) = \vec{v}(t)$ represents velocity and we can show that the velocity vector is always tangent to the path. Further, the second derivative $\vec{p}\,''(t) = \vec{v}\,'(t) = \vec{a}(t)$ is acceleration. To display the path determined by $\vec{p}(t)$ in Maple we will use `spacecurve`, whose argument is a three component expression in a single variable. For example, the output of the following lines is a spiral. Figure out why this is true.

```
> restart:   with(plots):
> spacecurve([cos(t),sin(t),t],t=0..2*Pi,color=red);
```

The next example is in fact an animation. The reader is encouraged to execute these lines, which will take some time because 800 frames must be constructed. Then, click on the resulting figure and observe that what looks like the display of a VCR appears above. Click on the triangle that suggests the "Play" button and watch what happens. After defining a position function, p, we differentiate it and make a new function v which represents the velocity. Differentiating again and making a function with `unapply`, we obtain our acceleration. The remaining lines set up the animation. The command `evalm` is used om vector expressions so as to simplify them into three component vectors. The output has been removed from the worksheet for simplicity.

Animation of position, velocity, and acceleration vectors

```
>  restart:   with(plots):   with(linalg):
> p:=t->[cos(t)+1,sin(t)+1,5*cos(2*t)+1];
> v:=unapply(diff(p(t),t),t);
> a:=unapply(diff(v(t),t),t);
```

```
> n:=200; nstart:=0; frameno:=n;  delta:=2*Pi/n;
> framenumbers:=[seq(nstart+i,i=1..(frameno))]:
```

The path is magenta, the position vector is blue, the velocity vector is red, and the acceleration vector is green. Note how the end of the position vector lies on the path and the velocity vector is tangent to it.

```
> A:=display(seq(spacecurve(evalm(s*p((i-1)*delta)),s=0..1,color=blue,thickness=2),
  i=framenumbers),insequence=true):
> B:=display(seq(spacecurve(evalm(s*v((i-1)*delta)+p((i-1)*delta)),s=0..1,color=red,
  thickness=2),i=framenumbers),insequence=true):

> C:=display(seq(spacecurve(evalm(s*a((i-1)*delta)+p((i-1)*delta)),s=0..1,color=green,
  thickness=2),i=framenumbers),insequence=true):
> P:=display(seq(spacecurve(p(t),t=0..2*Pi,color=magenta),i=framenumbers),
  insequence=true):
>  display(A,B,C,P,axes=normal,orientation=[75,45]);
```

We will begin by solving a ballistics problem analytically and then use Maple to solve it again. Assume that $\overrightarrow{p}(t)$ is the vector function that describes the position of the projectile at any time $t \geq 0$. We know that $\overrightarrow{p}\ \ (t)$ yields the velocity of the projectile and $\overrightarrow{p}\ \ \ (t)$ the acceleration. Further, we assume (falsely) that the only force affecting the projectile is gravity. Since gravity acts downward, our acceleration is the vector $-32\overrightarrow{j}$. We need two pieces of information about the projectile: (1) where is it at $t = 0$? and (2) what is the velocity at $t = 0$?, in order to solve this problem. What does it mean to 'solve' this problem, you may well ask? The objective is to find $\overrightarrow{p}(t)$ for $t \geq 0$. It is the basic process of starting with $\overrightarrow{p}\ \ \ (t)$ and working 'backwards' that is being taught here.

When information about the velocity of the projectile relative to the airplane is known and the velocity of the airplane is known, how does one find the velocity of the projectile. Verbally, the answer is "The velocity of the projectile is the velocity of the airplane plus the velocity of the projectile relative to the airplane." Mathematically, it looks like

$$\overrightarrow{V}_P = \overrightarrow{V}_A + \overrightarrow{V}_{P/A}$$

Vector Example: While flying 384 feet high over level ground at 300 ft/sec, a projectile is launched from an airplane. The muzzle velocity is 200 ft/sec in the direction of $\overrightarrow{u} = .6\overrightarrow{i} + .8\overrightarrow{j}$.

(a) With the point on the ground below as reference and a gravitational constant of 32 ft/sec/sec, find a position function $\overrightarrow{p}(t)$ that determines the position of the projectile at any time t.

We note first that $\overrightarrow{u}$ is a unit vector. The velocity of the airplane is $\overrightarrow{V}_A = 300\overrightarrow{i}$. The velocity of the projectile *relative to the airplane* is $\overrightarrow{V}_{P/A} = 200\overrightarrow{u} = 200(.6\overrightarrow{i} + .8\overrightarrow{j}) = 120\overrightarrow{i} + 160\overrightarrow{j}$. Thus the initial velocity of the projectile is

$$\overrightarrow{p}\ \ (0) = \overrightarrow{V}_P = \overrightarrow{V}_A + \overrightarrow{V}_{P/A} = 300\overrightarrow{i} + 120\overrightarrow{i} + 160\overrightarrow{j} = 420\overrightarrow{i} + 160\overrightarrow{j}$$

The initial position of the projection is $\overrightarrow{p}(0) = 384\overrightarrow{j}$.

We begin with the vector equation $\overrightarrow{p}\ \ \ (t) = \overrightarrow{a}(t) = -32\overrightarrow{j}$. Antidifferentiating we get $\overrightarrow{p}\ \ (t) = \overrightarrow{v}(t) = -32t\overrightarrow{j} + \overrightarrow{c}$. Because $\overrightarrow{p}\ \ (0) = 420\overrightarrow{i} + 160\overrightarrow{j}$, and $\overrightarrow{p}\ \ (0) = \overrightarrow{c}$, we have

$$\overrightarrow{p}\ \ (t) = -32t\overrightarrow{j} + 420\overrightarrow{i} + 160\overrightarrow{j} = 420\overrightarrow{i} + (160 - 32t)\overrightarrow{j}$$

Antidifferentiating again, we get $\overrightarrow{p}(t) = 420t\overrightarrow{i} + (160\,t - 16\,t^2)\overrightarrow{j} + \overrightarrow{c}_1$.
But $\overrightarrow{p}(0) = 384\,\overrightarrow{j}$, so we have our position function

$$\overrightarrow{p}(t) = 420t\overrightarrow{i} + (384 + 160t - 16t^2)\overrightarrow{j}$$

(b) When does the projectile land?

With $y(t)$ as the vertical component function, we see that we must have $y(t) = 0$.

$$\begin{aligned} y(t) = 384 + 160t - 16t^2 &= 0 \\ -16(t^2 - 10t - 24) &= 0 \\ (t-12)(t+2) &= 0 \\ t &= 12 \end{aligned}$$

(c) Where does it land?

The horizontal component function is $x(t) = 420t$. So the projectile lands $420 \cdot t \ = 420 \cdot 12 = 5040$ feet from the reference point.

(d) What is the speed at impact?

We must evaluate $\vec{v}(t\ ) = \vec{v}(12) = 420\vec{i} + (160 - 32 \cdot 12)\vec{j} = 420\vec{i} - 224\vec{j}$. Because speed is the length of the velocity vector, we have an impact speed in ft/sec of

$||\vec{v}(12)|| = ||420\vec{i} - 224\vec{j}|| = \sqrt{420^2 + 224^2} = 476$

(e) When does it reach maximum height?

We must maximize $y(t)$, so set $y\ (t) = 0$. This means that $160 - 32t = 0$ or $t = 5$seconds. Evaluating $y(t)$ at $t = 5$ yields a height of $384 + 160 \cdot 5 - 16 \cdot 25 = 784$ feet.

Before we begin the Maple solution it is important to understand that we cannot take a vector function such as $\langle f(t), g(t)\rangle$ in Maple and just integrate it. Note the syntax of the lines where `map` occurs. It allows the operation `int` to operate on each component. Further, when we substitute into such a function `op` allows access to each component of the vector function. If `vec1` is a vector, then `vec1[1]` refers to the first component of `vec1` and `vec1[2]` refers to the second component. The expressions `lhs, rhs` isolate the left hand side and right hand side respectively of an equation. Please take a good look at the operations in this section, because when we use them later we will assume that they are known to you.

Maple Example Use Maple to solve the projectile problem in the Vector Example.

(a) Find $\vec{p}(t)$.

```
> restart:   with(student):   with(linalg):
> acc:=[0,-32]:  va:=[300,0]:  p0:=[0,384]:
> u:=[3/5,4/5]:
> vpa:=evalm(200*u);
```

$$vpa := [120, 160]$$

```
> vp0:=evalm(va+vpa);
```

$$vp0 := [420, 160]$$

Set $\vec{p}\ \ (t) = -32\vec{j} =$ acc.

```
> p2prime:=evalm(acc);
```

$$p2prime := [0, -32]$$

We will need vectors as constants of integration.

```
> con1:=[c1,c2]:  con2:=[c3,c4]:
> p1prime:=evalm(map(int,p2prime,t)+con1);
```

$$p1prime := [c1, c2 - 32\,t]$$

```
> eq1:=evalm(subs(t=0,op(p1prime))=vp0);
```

$$eq1 := [c1, c2] = [420, 160]$$

```
> p1prime:=subs(c1=rhs(eq1)[1],c2=rhs(eq1)[2],op(p1prime));
```

$$p1prime := [420, 160 - 32\,t]$$

```
> p:=evalm(map(int,p1prime,t)+con2);
```

$$p := [420\,t + c3, 160\,t - 16\,t^2 + c4]$$

```
> eq2:=evalm(subs(t=0,op(p))=p0);
```

$$eq2 := [c3, c4] = [0, 384]$$

```
> p:=subs(c3=rhs(eq2)[1],c4=rhs(eq2)[2],op(p));
```

$$p := [420\,t, 160\,t - 16\,t^2 + 384]$$

And we have found $\vec{p}(t)$.

(b) When does the projectile land? Set the second component of $\vec{p}(t)$ equal to 0.

```
> eq3:=p[2]=0;
```

$$eq3 := -16t^2 + 160t + 384 = 0$$

```
> ans:=solve(eq3,t);
```

$$ans := -2, 12$$

(c) Where does it land? Note how we access the second answer, which is the time it lands.

```
> lands:=subs(t=ans[2],op(p));
```

$$lands := [5040, 0]$$

(d) How fast was it going at impact? We must go back to $\vec{p}\ (t)$.

```
> vel:=subs(t=ans[2],op(p1prime));
```

$$vel := [420, -224]$$

```
> speed:=norm(vel,2);
```

$$speed := 476$$

(e) How high does the projectile go? We must maximize the second component.

```
> eq4:=p1prime[2]=0;
```

$$eq4 := -32t + 160 = 0$$

```
> t1:=solve(eq4,t);
```

$$t1 := 5$$

```
> height:=subs(t=t1,p[2]);
```

$$height := 784$$

C3M3 Problem Use Maple to solve the problem below.

A plane flies in low and ascends at 600 ft/sec in the direction of $.8\vec{i} + .6\vec{j}$ where $\vec{i}$, $\vec{j}$ have the usual orientations. At the time that the plane is $12,000$ feet high, a projectile is fired with a muzzle velocity of 400 ft/sec in the direction of $\frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j}$. Assume that the gravitational constant $g = 32\ ft/sec^2$.

(a) Find a position function $\vec{p}(t)$ for the projectile for any time $t \geq 0$. Let the point on the ground below the plane when it fires the projectile be the reference point.

(b) Find the time t when the projectile strikes the ground.

(c) Find the distance from the reference point to the point of impact.

(d) Find the speed of the projectile at the time of impact.

(i) in feet per second

(ii) in miles per hour. Recall that 60 miles per hour is 88 feet per second.

(e) Find the maximum height attained by the projectile and the time it occurs.

C3M4

Parametric Surfaces

To parameterize a surface we must do two things. We must determine a portion of the plane to serve as the domain of a continuous function. Then we must determine that function that maps this domain onto the surface we are parameterizing in a nice way. Later we will learn why this is of value. You will find that we will lean heavily on $z = f(x, y)$, $y = g(x, z)$, or $x = h(y, z)$ in the rectangular case. But cylindrical or spherical coordinates will be of equal value in accomplishing this task.

If D is a set in $\mathbb{R}^2$ and g is defined as

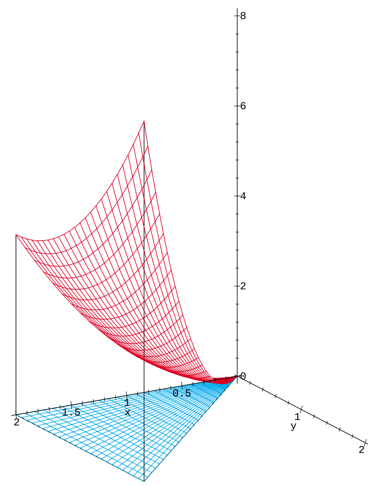
$$g\colon D \to \mathbb{R}^3 \qquad g(u,v) = \begin{matrix} x \\ y \\ z \end{matrix} \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \\ g_3(u,v) \end{pmatrix}$$

then our surface is defined as the image $g(D) = S$. This looks worse than it really is. For one thing, the x, y, z outside of the parentheses do not normally go there. They were put there this time to emphasize that the entry for that row determines the value of the x, y, z coordinate.

Suppose $z = f(x,y)$ with domain D. Define $g(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$ where it is obvious that u and v play the role of x and y. Or, just use x and y as the independent variables. After a few examples this will seem easier. Note how the parameterization ties in nicely with the plotting of the surface. While we could easily use *cylinderplot* or *sphereplot* in certain problems, having the function g makes it much easier to just use *plot3d*. One of the tricks to ease parameterization is to ask yourself this question: "Is the surface constant for any of the variables in any of the three coordinate systems that we use?" If so, using the other two variables is probably the easiest way to proceed.

In each example that follows we will define the function in Maple and then use *plot3d* to display it. The first step makes us get used to the idea that we are defining a function and the second forces us to define the domain of the function. When we set up the *plot3d* restrictions on the variables we are defining the domain. These skills will be essential later in the course.

Example 1 Parameterize the portion of the surface $z = x^2 + y^2$ that lies above the triangle with vertices $P(0,0)$, $Q(2,0)$, $R(2,2)$. Because $z = f(x,y)$, we just use the obvious approach as described above.



$$g(x,y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix} \qquad \begin{array}{c} 0 \le y \le x \\ 0 \le x \le 2 \end{array}$$

Define the function in Maple:

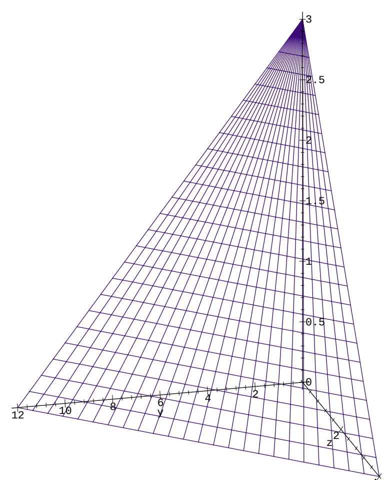
```
> g:=(x,y)->[x,y,x^2+y^2];
```

$$g := (x,y) \to [x, y, x^2 + y^2]$$

To plot this surface use:

```
> plot3d(g(x,y),x=0..2,y=0..x,color=red);
```

Example 2 Parameterize that portion of the plane $x + 3y + 4z = 12$ that lies in the first octant with x as the dependent variable.



$$g(y,z) = \begin{pmatrix} 12 - 3y - 4z \\ y \\ z \end{pmatrix} \qquad \begin{array}{c} 0 \le y \le 4 - \frac{4z}{3} \\ 0 \le z \le 3 \end{array}$$

Define the function in Maple:

```
> g:=(y,z)->[12-3*y-4*z,y,z];
```

$$g := (y,z) \to [12 - 3y - 4z, y, z]$$

And to plot this surface:

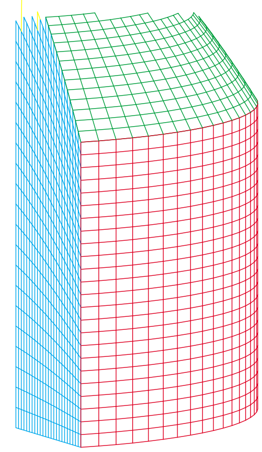
```
> plot3d(g(y,z),y=0..4-4*z/3,z=0..3,color=blue);
```

You may be puzzled by this plot. Pay attention to the axes. Because the bounds on y were listed first, Maple thinks that y belongs where we put the x-axis. It is very important that you realize that the domain of this function g is the triangle in the yz-plane bounded by the y and z axes and the plane $x+3y+4z=12$.

Before we do this next example we remind you of the cylindrical coordinate system.

$$\begin{matrix} x \\ y \\ z \end{matrix} \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Example 3 The solid in the first octant lies between $x^2+y^2=1$ and $x^2+y^2=4$, above $z=0$ and below the paraboloid $z=9-x^2-y^2$. We are going to parameterize the surfaces that we can see and show how to plot them in Maple. This is easiest when done from the viewpoint of cylindrical coordinates. We will begin with the surface on the left, where $y=0$ or $\theta=0$. Note that having $\theta=0$ lets us use r and z. Observe that $\cos 0=1$ and $\sin 0=0$.



$$g1(r,z)=\begin{pmatrix} r \\ 0 \\ z \end{pmatrix} \qquad \begin{matrix} 1\le r\le 2 \\ 0\le z\le 9-r^2 \end{matrix}$$

The function that parameterizes the top is:

$$g2(r,\theta)=\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 9-r^2 \end{pmatrix} \qquad \begin{matrix} 1\le r\le 2 \\ 0\le \theta\le \pi/2 \end{matrix}$$

The domain is the annular region between the circles in the first quadrant.

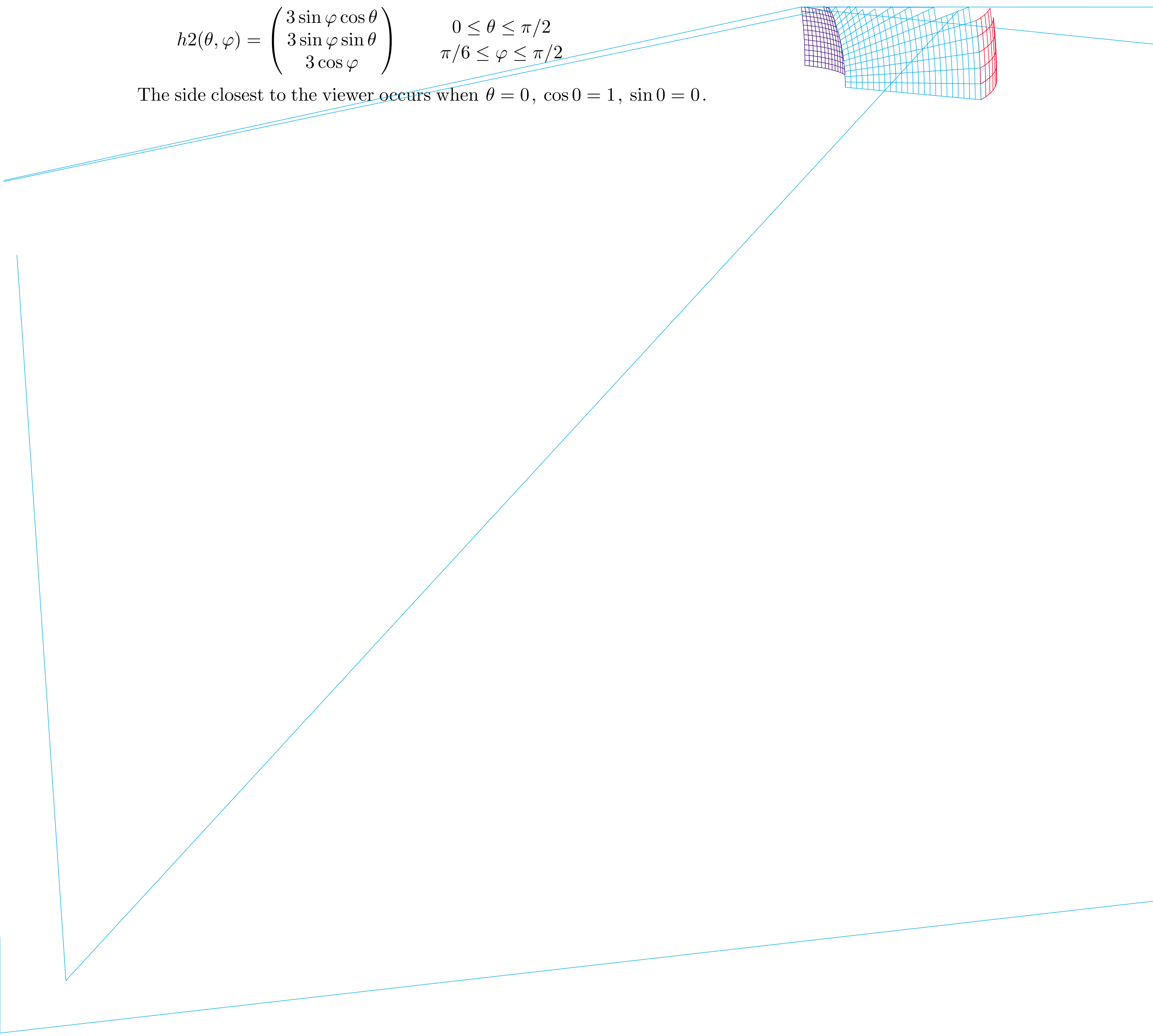
Example 4 The solid in the first octant lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$, $(x \geq 0, y \geq 0, z \geq 0)$ and below the cone $z = \sqrt{3(x^2 + y^2)}$. We are going to parameterize the four surfaces that we can see in the figure, beginning with the small spherical surface to the lower left where $\rho = 1$. The cone on the top is produced by letting $\varphi = \pi/6$.

$$h1(\theta, \varphi) = \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix} \qquad \begin{matrix} 0 \leq \theta \leq \pi/2 \\ \pi/6 \leq \varphi \leq \pi/2 \end{matrix}$$

By adjusting ρ to be 3 we have the outside spherical surface.

$$h2(\theta, \varphi) = \begin{pmatrix} 3\sin\varphi\cos\theta \\ 3\sin\varphi\sin\theta \\ 3\cos\varphi \end{pmatrix} \qquad \begin{matrix} 0 \leq \theta \leq \pi/2 \\ \pi/6 \leq \varphi \leq \pi/2 \end{matrix}$$

The side closest to the viewer occurs when $\theta = 0$, $\cos 0 = 1$, $\sin 0 = 0$.



C3M5a

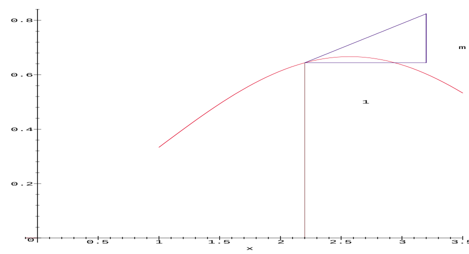
Tangent Planes

We will learn soon that if f is a real-valued function with domain $D \subseteq \mathbb{R}^2$, i.e. $D \xrightarrow{f} \mathbb{R}$ then the gradient of f at $X = (x, y)$ denoted by $\nabla f(X)$, $grad\, f(X)$, or $grad\, f\big|_X$ is defined by

$$\nabla f(X) = \left\langle \frac{\partial f}{\partial x}\bigg|_X , \frac{\partial f}{\partial y}\bigg|_X \right\rangle = \big\langle f_x(x,y), f_y(x,y)\big\rangle = f_x(x,y)\overrightarrow{i} + f_y(x,y)\overrightarrow{j}$$

The gradient is a very important concept that is useful when discussing rates of change of functions of several variables. It is usually introduced in a course after partial derivatives have been defined, the chain rule has been discussed, and when the applications are being surveyed. Once the concept of the derivative of a real-valued multivariable function is introduced, the student can understand that this derivative somehow coincides with the gradient. The actual derivative at a point is a *linear mapping* which is evaluated by multiplication by a matrix. And that matrix is none other than the gradient (with the commas deleted).

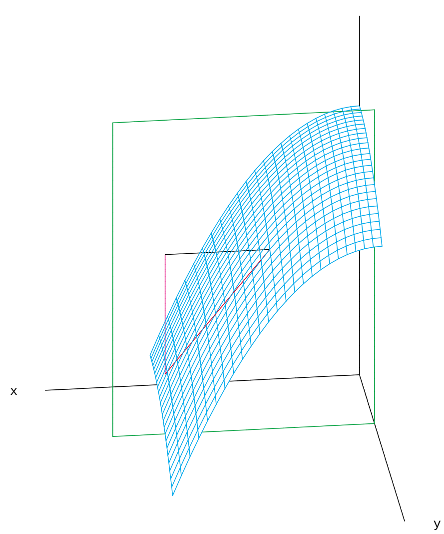
In the single variable case where $y = f(x)$ and x_0 is a point in the domain, if $m = f\,(x_0)$, then $\langle m \rangle$ is the gradient. The slope of the tangent line at x_0 is m and if one moves one unit to the right of x_0, then the change in value for the tangent line is exactly m. So this simple diagram is valid.

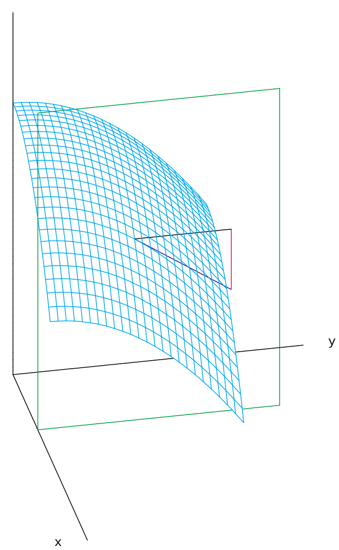


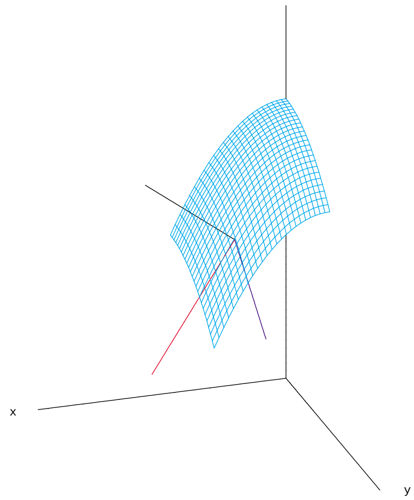
Returning to the case where $z = f(x, y)$, the same geometric approach is valid. When (x_0, y_0) is in the domain of f and the graph of f is viewed in $\mathbb{R}^3$, consider the vertical planes containing $(x_0, y_0, 0)$ that are parallel to the xz-plane and the yz-plane. Let's put this on a more concrete footing. Suppose that

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} = f_x(x_0, y_0) = m_1 \qquad \text{and} \qquad \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} = f_y(x_0, y_0) = m_2$$

While it is true that $\nabla f(x_0, y_0) = \langle m_1, m_2 \rangle$, our focus here is not on the gradient. Rather, for now it is sufficient that one understand that we have evaluated the partial derivatives at (x_0, y_0) and obtained two **numbers** that serve as slopes in the x and y directions respectively. Then, we construct the vectors $\overrightarrow{v_1} = \langle 1, 0, m_1 \rangle$ and $\overrightarrow{v_2} = \langle 0, 1, m_2 \rangle$ and note that each is a **tangent vector** to the surface Σ defined by $z = f(x, y)$ when they are viewed as emanating from $\overrightarrow{X_0} = \langle x_0, y_0, f(x_0, y_0) \rangle$. It is very important to have a geometric understanding of where $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ fit in the picture. We see that $\overrightarrow{v_1}$ is the tangent vector in the vertical plane parallel to the xz-plane and $\overrightarrow{v_2}$ is the tangent vector in the vertical plane parallel to the yz-plane. Each is aimed downward here. Observe the diagrams that follow.







For purposes of finding the basic equation for a plane tangent to the surface Σ, it is best to use $-\overrightarrow{N} = \overrightarrow{N_1} = \langle m_1, m_2, -1 \rangle$. If $\overrightarrow{X} = \langle x, y, z \rangle$

$$\overrightarrow{N_1} \cdot \left(\overrightarrow{X} - \overrightarrow{X_0} \right) = 0$$

$$\Longrightarrow \quad m_1(x - x_0) + m_2(y - y_0) - (z - z_0) = 0$$

$$\Longrightarrow \quad z - z_0 = m_1(x - x_0) + m_2(y - y_0)$$

yields three forms of our equation for the tangent plane. It is important to remember that m_1 and m_2 are numbers and are **NOT** expressions including variables.

Maple Example: For $f(x,y) = (9 - 2x^2 - y^2)/3$ and $(x_0, y_0) = (1,1)$, find:

(a) a vector normal to the surface defined by $z = f(x,y)$ at $(1,1)$, and an equation for the tangent plane at $X_0 = \big(x_0, y_0, f(x_0, y_0)\big)$.

(b) Plot the surface, normal vector, tangent plane, and a line from $(x_0, y_0, 0)$ to X_0.

```
> restart:   with(plots):   with(linalg):
> f:=(x,y)->(9-2*x^2-y^2)/3;
```

$$f := (x,y) \to 3 - \frac{2}{3}x^2 - \frac{1}{3}y^2$$

```
> x0:=1;  y0:=1;  z0:=f(x0,y0);
```

$$x0 := 1$$

$$y0 := 1$$

$$z0 := 2$$

```
> X:=[x,y,z];  X0:=[x0,y0,z0];
```

$$X := [x, y, z]$$

$$X0 := [1, 1, 2]$$

```
> fx:=diff(f(x,y),x);  fy:=diff(f(x,y),y);
```

$$fx := -\frac{4}{3}x$$

$$fy := -\frac{2}{3}y$$

```
> m1:=subs(x=x0,y=y0,fx);  m2:=subs(x=x0,y=y0,fy);
```

$$m1 := \frac{-4}{3}$$

$$m2 := \frac{-2}{3}$$

Our tangent vectors in the x and y directions are determined.

```
> vx:=[1,0,m1];  vy:=[0,1,m2];
```

$$vx := \left[1, 0, \frac{-4}{3}\right]$$
$$vy := \left[0, 1, \frac{-2}{3}\right]$$

We find the normal vector, which is orthogonal to our two tangent vectors.

```
> N:=crossprod(vx,vy);
```

$$N := \left[\frac{4}{3}, \frac{2}{3}, 1\right]$$

```
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> zaxis:=spacecurve([0,0,t],t=0..4,color=black):
> surff:=plot3d(f(x,y),x=0..2,y=0..2,color=cyan):
```

The line which represents the normal vector to the surface is plotted using a vector expression. We use 'evalm' to evaluate the expression to obtain the vector format we need for 'spacecurve'.

```
> Nline:=spacecurve(evalm(X0+t*N),t=0..1,color=magenta):
> X1:=vector([x0,y0,0]);
```

$$X1 := [x0, y0, 0]$$

Important: The easiest way to parameterize a line segment between two points (or vectors) P and Q is as $(1-t)P + tQ$ for $0 \le t \le 1$. We apply this in our next plot.

```
> Vline:=spacecurve(evalm((1-t)*X1+t*X0),t=0..1,color=blue):
```

Use the basic equation $\overrightarrow{N} \cdot \overrightarrow{X} = \overrightarrow{N} \cdot \overrightarrow{X_0}$ for a tangent plane to get an expression for z.

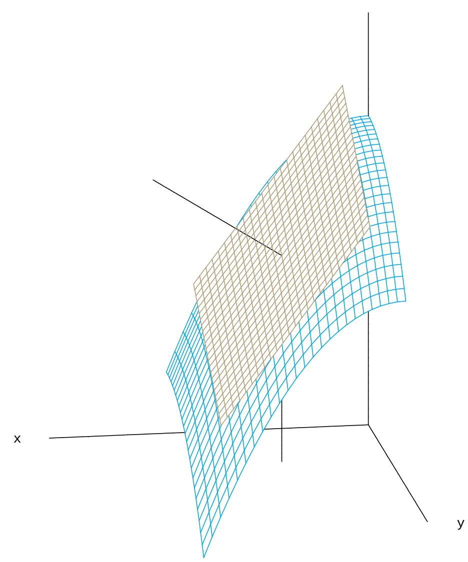
```
> eq1:=innerprod(N,X)=innerprod(N,X0);
```

$$eq1 := \frac{4}{3}x + \frac{2}{3}y + z = 4$$

```
> zee:=solve(eq1,z);
```

$$zee := -\frac{4}{3}x - \frac{2}{3}y + 4$$

```
> tplane:=plot3d(zee,x=.3..(1.7),y=.3..(1.7),color=sienna):
> display(xaxis,yaxis,zaxis,surff,Nline,Vline,tplane);
```



C3M5a Problem: Given $f(x, y) = 2\cos(x) + 2\sin(x)\cos(y)$ and $(x_0, y_0) = (\pi/3, -\pi/3)$. Use Maple to:

(a) Find an equation of the tangent plane to the surface $z = f(x, y)$ at $\big(x_0, y_0, f(x_0, y_0)\big)$.

(b) Plot the surface for $0 \le x \le 1.8$ and $-1.5 \le y \le .5$, a line representing the normal vector at $\big(x_0, y_0, f(x_0, y_0)\big)$, a line from $(x_0, y_0, 0)$ to $\big(x_0, y_0, f(x_0, y_0)\big)$. Include the coordinate axes. Hint: let the

y-axis range from -2 to 2.

C3M5b

The Gradient and Tangent Planes for Level Surfaces of $G(x, y, z) = c$

Suppose that $D \subseteq \mathbb{R}^3$ is the domain of the real-valued function G. That is, $D \xrightarrow{G} \mathbb{R}$. If L_c is the subset of D for which $G(x, y, z) = c$, then L_c is a level surface of or for G for the constant c. For example, if $G(x, y, z) = x^2 + y^2 + z^2$ and $c = 9$, then L_9 is the sphere of radius 3 centered at the origin. Different values of c produce different spheres as level surfaces of G. Thus, for a given function G and value c we may define a surface Σ as L_c. Now suppose that $\overrightarrow{\alpha}(t)$ is a space curve in Σ. That is, $[a, b] \xrightarrow{\overrightarrow{\alpha}} \Sigma$. Suppose $\overrightarrow{\alpha}(t) = \langle f(t), g(t), h(t)\rangle$.

By the nature of our assumptions, the composition $G\big(\overrightarrow{\alpha}(t)\big) = c$ for each t. The overall effect of our composition function $G \circ \overrightarrow{\alpha}$ is that the function is a constant. It only assumes one value, c. The derivative of G is represented by a 1×3 matrix that varies over the domain.

$$DG(\overrightarrow{\alpha}(t)) = [\, G_x\big(\overrightarrow{\alpha}(t)\big) \quad G_y\big(\overrightarrow{\alpha}(t)\big) \quad G_z\big(\overrightarrow{\alpha}(t)\big)\,]$$

Now let's apply the chain rule to $G\big(\overrightarrow{\alpha}(t)\big)$ as we differentiate with respect to t. The stars on the first and second lines are used to represent matrix multiplication.

$$\begin{aligned} D_t\big(G(\overrightarrow{\alpha}(t))\big) &= DG(\overrightarrow{\alpha}(t)) \star \overrightarrow{\alpha}\ (t) = 0 \\ [\, G_x\big(\overrightarrow{\alpha}(t)\big) \quad G_y\big(\overrightarrow{\alpha}(t)\big) \quad G_z\big(\overrightarrow{\alpha}(t)\big)\,] \star \begin{bmatrix} f\ (t) \\ g\ (t) \\ h\ (t) \end{bmatrix} &= 0 \\ [G_x\big(\overrightarrow{\alpha}(t)\big) f\ (t) + G_y\big(\overrightarrow{\alpha}(t)\big) g\ (t) + G_z\big(\overrightarrow{\alpha}(t)\big) h\ (t)] &= 0 \\ \Longrightarrow \quad \nabla G\big(\overrightarrow{\alpha}(t)\big) \cdot \overrightarrow{\alpha}\ (t) &= 0 \end{aligned}$$

Conclusion: The gradient at X_0, $\nabla G(X_0)$, is orthogonal to every tangent vector (at X_0) of the level surface of G that contains X_0. The gradient vector at a point is normal to the level surface of G that contains the point.

This makes it easy to find the equation of a tangent plane to a level surface for the point where $t = t_0$. If $\nabla G(\overrightarrow{\alpha}(t_0)) = \langle m_1, m_2, m_3\rangle$, $\overrightarrow{\alpha}(t_0) = \langle x_0, y_0, z_0\rangle = \overrightarrow{X_0}$,and $\overrightarrow{X} = \langle x, y, z\rangle$, then the equation of the tangent plane becomes

$$\begin{aligned} \nabla G(\overrightarrow{\alpha}(t_0)) \cdot \big(\overrightarrow{X} - \overrightarrow{X_0}\big) &= 0 \\ \langle m_1, m_2, m_3\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle &= 0 \\ \text{or} \quad m_1(x - x_0) + m_2(y - y_0) + m_3(z - z_0) &= 0 \end{aligned}$$

Example: It is important to see the consistency between this section and the case where $z = f(x, y)$. Let G be defined as $G(x, y, z) = f(x, y) - z$ and suppose $z_0 = f(x_0, y_0)$, $f_x(x_0, y_0) = m_1$, $f_y(x_0, y_0) = m_2$, and $X_0 = (x_0, y_0, z_0)$. Then

$$\begin{aligned} \nabla G(X_0) &= \langle G_x(X_0),\, G_y(X_0),\, G_z(X_0)\rangle \\ &= \langle f_x(x_0, y_0),\, f_y(x_0, y_0),\, -1\rangle = \langle m_1, m_2, -1\rangle \end{aligned}$$

The vector $\langle m_1, m_2, -1\rangle = \overrightarrow{N_1}$ is exactly the normal vector that we used to establish the equation for the tangent plane in the previous section and we repeat that process here. The equation for the tangent plane is

$$\begin{aligned} \overrightarrow{N_1} \cdot \left(\overrightarrow{X} - \overrightarrow{X_0}\right) &= 0 \\ \Longrightarrow \quad m_1(x - x_0) + m_2(y - y_0) - (z - z_0) &= 0 \\ \Longrightarrow \quad z - z_0 &= m_1(x - x_0) + m_2(y - y_0) \end{aligned}$$

Maple Example: Use Maple to find an equation of the plane tangent to the level surface of $G(x, y, z) = (2x^2 + 4y^2 + 6z^2 + xy + yz)/36$ that contains $X_0 = (1, 2, 3)$. Plot the surface near X_0, the tangent plane, and the gradient.

```
> with(linalg):  with(plots):  with(student):
> x0:=1:    y0:=2:    z0:=3:    X0:=vector([x0,y0,z0]);
```

$$X0 := [1, 2, 3])$$

```
> G:=(x,y,z)->(2*x^2+4*y^2+6*z^2+x*y+y*z)/36;
```

$$G := (x, y, z) \to \frac{1}{18}x^2 + \frac{1}{9}y^2 + \frac{1}{6}z^2 + \frac{1}{36}xy + \frac{1}{36}yz$$

```
> K:=G(x0,y0,z0);
```

$$K := \frac{20}{9}$$

```
> gradG:=grad(G(x,y,z),[x,y,z]);
```

$$gradG := \left[\frac{1}{9}x + \frac{1}{36}y, \frac{2}{9}y + \frac{1}{36}x + \frac{1}{36}z, \frac{1}{3}z + \frac{1}{36}y\right]$$

```
> N:=subs(x=x0,y=y0,z=z0,op(gradG));
```

$$N := \left[\frac{1}{6}, \frac{5}{9}, \frac{19}{18}\right]$$

```
> X:=vector([x,y,z]):
> tplane1:=evalm(innerprod(N,X)=innerprod(N,X0));
```

$$tplane1 := \frac{1}{6}x + \frac{5}{9}y + \frac{19}{18}z = \frac{40}{9}$$

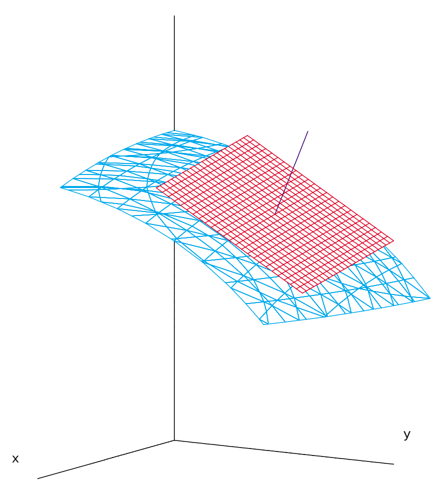
Obviously, *tplane1* is an equation for the tangent plane at X_0.

```
> zee:=solve(tplane1,z);
```

$$zee := -\frac{3}{19}x - \frac{10}{19}y + \frac{80}{19}$$

Now let's set up the plots, including labels for the axes.

```
> tplane:=plot3d(zee,x=0..2,y=1..3,color=red):
> graphG:=implicitplot3d(G(x,y,z)=K,x=0..2,y=0..3,z=2..4,color=cyan):
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> zaxis:=spacecurve([0,0,t],t=0..5,color=black):
> Nvec:=spacecurve(evalm(X0+t*N),t=0..1,color=blue):
> xlabel:=textplot3d([3,-.3,.2,"x"],color=black):
> ylabel:=textplot3d([-.3,3,.3,"y"],color=black):
> display(tplane,graphG,xaxis,yaxis,zaxis,Nvec,xlabel,ylabel);
```



C3M5b Problem: Given: $G(x, y, z) = x^2 + y + z^2 - 3$ and $X_0 = (1, 1, 1)$.

(a) Use Maple to find the gradient of G at X_0 and an equation for the tangent plane to the level surface of G that contains X_0.

(b) Plot the level surface of G, the tangent plane at X_0, and a line that represents the gradient of G at X_0. Include coordinate axes.

Suggestions: For the surface, use `implicitplot3d` with $0 \le x \le 2.5$, $0 \le y \le 2.5$, and $0 \le z \le 2.5$. For the tangent plane, use `plot3d` with $.5 \le x \le 1.5$ and $.5 \le y \le 1.5$.

C3M6

Extrema of Functions of Two Variables

In this assignment we will be solving simultaneous equations and using matrices in order to locate possible relative maxima and minima. After defining the function f and determining the partial derivatives of f, we set those partials equal to zero and solve the equations. We obtain an unexpected "**RootOf**" in our first solution, so the example shows how to resolve that. The notation "**s[1]**" refers to the first set of brackets in the solution "**s**" and "**allvalues**" yields the two solutions where "**RootOf**" has occurred. At a point, such as (x_0, y_0), where both partial derivatives are zero we test to see if the determinant of the Hessian $H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ evaluated at (x_0, y_0) is positive or negative. The positive case yields a relative maximum or minimum, while the negative case indicates that (x_0, y_0) is a saddle point. We test further if the determinant of the Hessian is positive because this can only occur if the second partial derivatives, f_{xx} and f_{yy}, have the same sign and may be interpreted as indicating concave "up" or "down". Because the entry of the Hessian, H, in the $(1,1)$ position is f_{xx}, we simply evaluate $H[1,1]$ at the point in question and we have our answer. We will need three packages for Maple: student, linalg, and plots.

Example: Find all critical points of the function $f(x,y) = x^3 + 3xy^2 - 4y^3 - 15x$ and determine which, if any, are relative maxima or minima or saddle points. For simplicity, we will define f as an expression.

```
> restart:  with(student):  with(plots):  with(linalg):
> f:=x^3+3*x*y^2-4*y^3-15*x;
```

$$f := x^3 + 3xy^2 - 4y^3 - 15x$$

```
> fx:=diff(f,x);
```

$$fx := 3x^2 + 3y^2 - 15$$

```
> fy:=diff(f,y);
```

$$fy := 6xy - 12y^2$$

Solve the simultaneous equations for x and y.

```
> s:=solve({fx=0,fy=0},{x,y});
```

$$s := \{y = 0, x = RootOf(\_Z^2 - 5)\}, \{y = 1, x = 2\}, \{x = -2, y = -1\}$$

```
> s1:=allvalues(s[1]);
```

$$s1 := \{y = 0, x = \sqrt{5}\}, \{y = 0, x = -\sqrt{5}\}$$

```
> H:=hessian(f,[x,y]);
```

$$H := \begin{bmatrix} 6x & 6y \\ 6y & 6x - 24y \end{bmatrix}$$

Insert the first of the solutions listed in "s1" into H. The use of "op" reminds Maple that H is a matrix and allows access to the components of H.

```
> H1:=subs(s1[1],op(H));
```

$$H1 := \begin{bmatrix} 6\sqrt{5} & 0 \\ 0 & 6\sqrt{5} \end{bmatrix}$$

```
> a1:=det(H1);
```

$$a1 := 180$$

```
> b1:=subs(s1[1],H[1,1]);
```

Testing the sign of f_{xx} or fxx at the point.

$$b1 := 6\sqrt{5}$$

Because the determinant of the Hessian is positive and b1 is positive (indicating 'concave up'), there is a relative minimum at `s1[1]`$= (\sqrt{5}, 0)$.

```
> v1:=subs(s1[1],f);
```

$$v1 := -10\sqrt{5}$$

```
> s2:=s1[2];
```

$$s2 := \{y = 0, x = -\sqrt{5}\}$$

This line was to demonstrate what '`s1[2]`' would produce.

```
> H2:=subs(s2,op(H));
```

$$H2 := \begin{bmatrix} -6\sqrt{5} & 0 \\ 0 & -6\sqrt{5} \end{bmatrix}$$

```
> a2:=det(H2);
```

$$a2 := 180$$

```
> b2:=subs(s2,H[1,1]);
```

$$b2 := -6\sqrt{5}$$

Because the Hessian is positive at s2 and b2 is negative (indicating 'concave down'), there is a relative maximum at s2$= (\sqrt{5}, 0)$. Now we compute the value of f at s2.

```
> v2:=subs(s2,f);
```

$$v2 := 10\sqrt{5}$$

```
> H3:=subs(s[2],op(H));
```

$$H3 := \begin{bmatrix} 12 & 6 \\ 6 & -12 \end{bmatrix}$$

's[2]' is the second set listed of the original set, 's'.

```
> a3:=det(H3);
```

$$a3 = -180$$

Because the determinant of the Hessian is negative, we have a saddle point at s[2]$= (2, 1)$.

```
> v3:=subs(s[2],f);
```

$$v3 := -20$$

```
> H4:=subs(s[3],op(H));
```

$$H4 := \begin{bmatrix} -12 & -6 \\ -6 & 12 \end{bmatrix}$$

```
> a4:=det(H4);
```

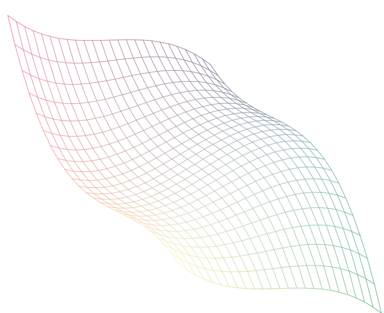
$$a4 := -180$$

```
> v4:=subs(s[3],f);
```

$$v4 := 20$$

Because the determinant of the Hessian is negative at s[3]$= (-2, -1)$, there is a saddle point there.
Now let's see what the graph of f looks like.

```
> plot3d(f,x=-6..6,y=-5..5,color=black);
```



```
> contourplot(f,x=-6..6,y=-4..4,contours=[-28,-24,-20,-16,-12,-8,-4,0,4,8,12,16,20,24,
28],color=black);
```

First, rewrite the integral and insert the variables of integration in the limits. $\int_{x=-2}^{x=1}\int_{y=\arctan(x)}^{y=e^x} y+x^2\,dy\,dx$

This tells us that for any fixed value of x, y must be able to vary from $\arctan(x)$ to e^x inside the domain R. For simplicity, let's define these functions in Maple, assuming that `student` and `plots` have been invoked.

```
> h:=x->exp(x);
```

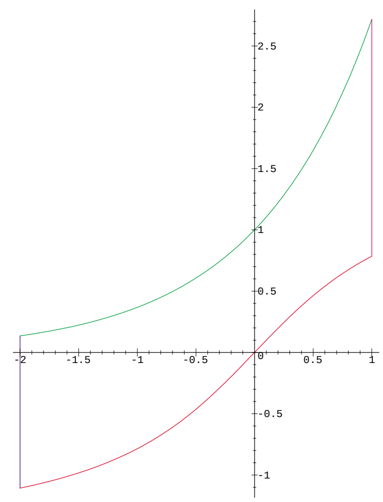
$$h := \exp$$

```
> g:=x->arctan(x);
```

$$g := \arctan$$

We will display four plots in parametric form that determine the boundaries of the region. Make sure you can identify each of them. What roles do the outer limits of our integral play? They determine the domains of g and h and occur here as vertical line segments, $x=-2$ and $x=1$. We display the output next to the commands so as to save space here.

```
> A1:=plot([x,g(x),x=-2..1],color=red):
> A2:=plot([x,h(x),x=-2..1],color=green):
> A3:=plot([-2,y,y=g(-2)..h(-2)],color=blue):
> A4:=plot([1,y,y=g(1)..h(1)],color=magenta):
> display(A1,A2,A3,A4);
```



Note how if a vertical line is moved from left to right on this plot it will always intersect the graph of g on the bottom and the graph of h on the top, i.e. curve-to-curve. What would happen if we moved a horizontal line from the bottom to the top of the plot? On the left it would intersect either the vertical line $x=-2$ or the graph of h, while on the right it would cut across either the graph of g or the vertical line $x=1$. This violates the principle of "curve-to-curve, point-to-point", which means that the reverse order of integration will not work for this region. You may be surprised to note that $f(x,y)=y+x^2$ does not play a role in the domain of the integral unless f is not defined at some point in R. In Maple the integral would be defined by:

```
> Doubleint(y+x^2,y=arctan(x)..exp(x),x=-2..1);
```

$$\int_{-2}^{1}\int_{\arctan(x)}^{e^x} y+x^2\,dy\,dx$$

Think about the syntax we used to plot the bottom and top curves above.

Example 2: Discuss $\int_0^1\int_{y^2}^{2-y} y+x^2\,dx\,dy$

As before, start with $\int_{y=0}^{y=1}\int_{x=y^2}^{x=2-y} y+x^2\,dx\,dy$ and define the inner limits as functions.

```
> r:=y->y^2;
```

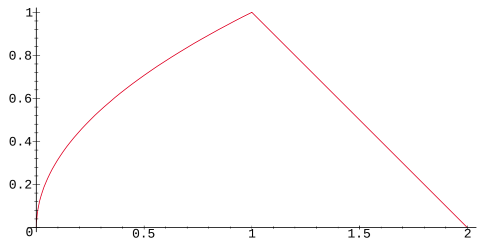
$$r := y \to y^2$$

```
> s:=y->2-y;
```

$$s := y \to 2-y$$

Now plot just these two curves:

```
> display(plot([r(y),y,y=0..1]),plot([s(y),y,y=0..1]));
```



Since we integrated first with respect to x, we must be able to draw horizontal lines that always begin on the left curve and end on the right curve. Sliding a horizontal line up the page, the first y value for which the line touches the region is $y = 0$ and the last value is $y = 1$.

C3M7 Problems: Plot the domains of the double integrals listed.

1. $\displaystyle\int_0^1 \int_{x^2}^{2x} f(x,y)\,dy\,dx$

2. $\displaystyle\int_0^2 \int_{-y}^{y^2} f(x,y)\,dx\,dy$

3. $\displaystyle\int_0^1 \int_{y}^{2-y} f(x,y)\,dx\,dy$

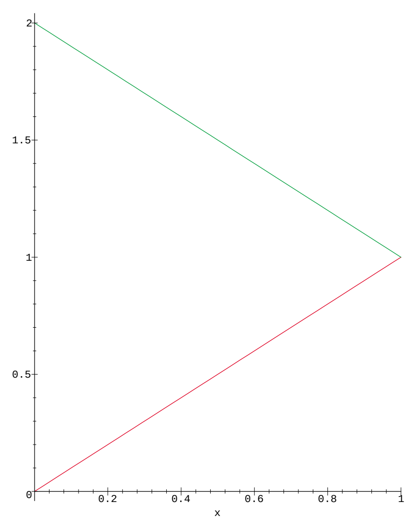
4. $\displaystyle\int_{-1}^2 \int_{x^2}^{2+x} f(x,y)\,dy\,dx$

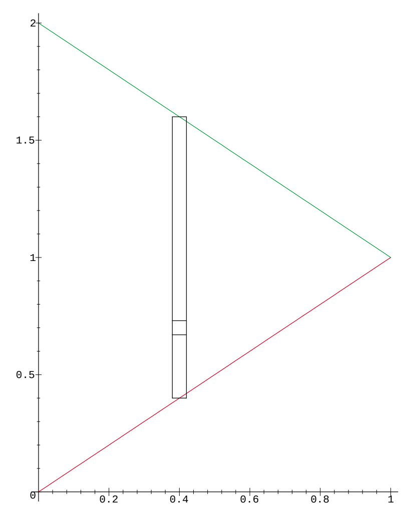
C3M8

Setting up Double Integrals

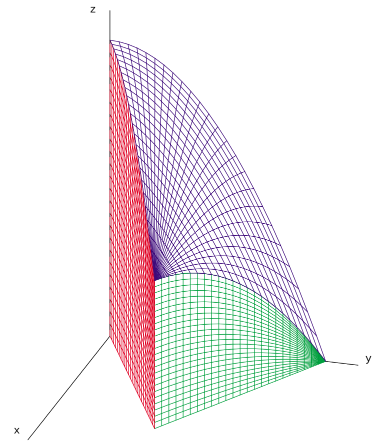
We will begin by drawing a region and showing how the geometry of the region leads to the set-up of an iterated integral. The operative phrase is “curve-to-curve, point-to-point”. As we promised in an earlier session, we will begin with a region that is bounded by $y = x,\ y = 2 - x,\ x = 0$. The Maple syntax shown will produce the plot on the left. The right-hand plot is provided to help explain why the region requires that $dA = dy\,dx$.

```
> restart:    with(plots):
> plot([x,2-x],x=0..1);
```

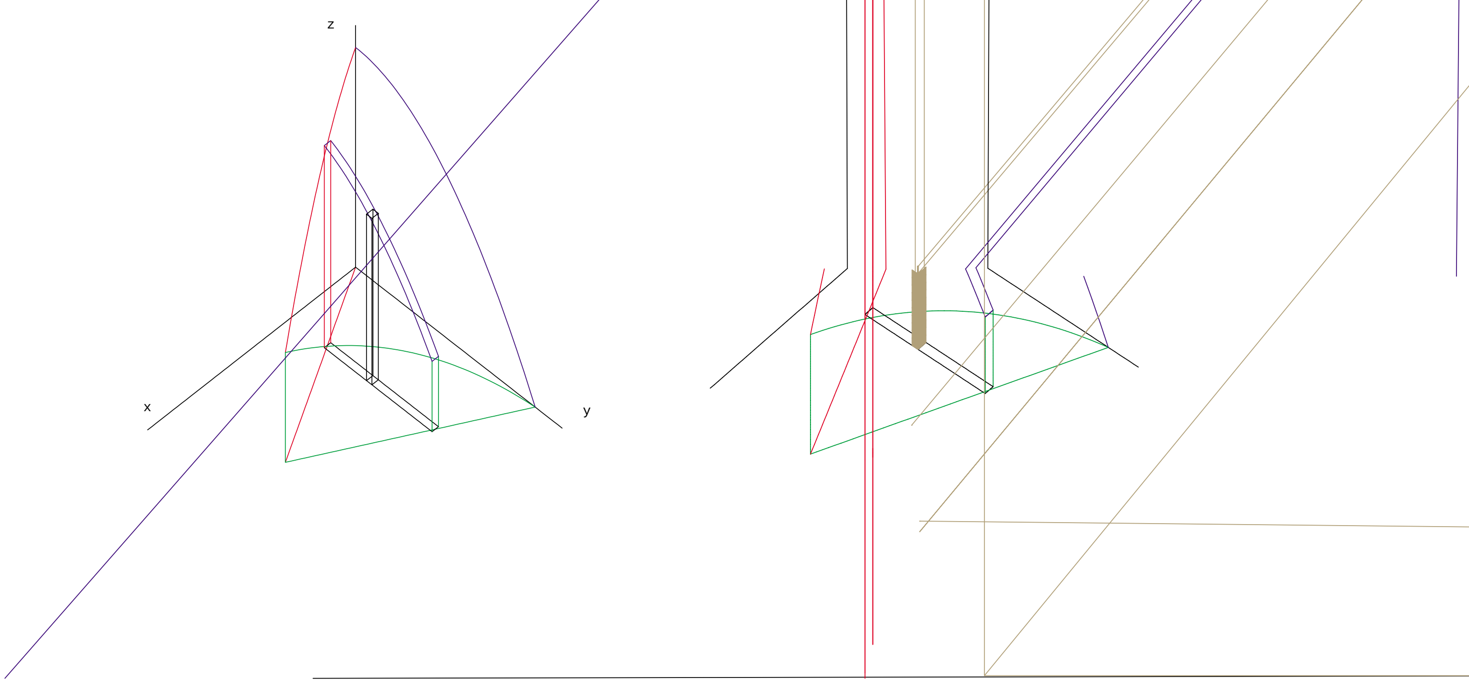




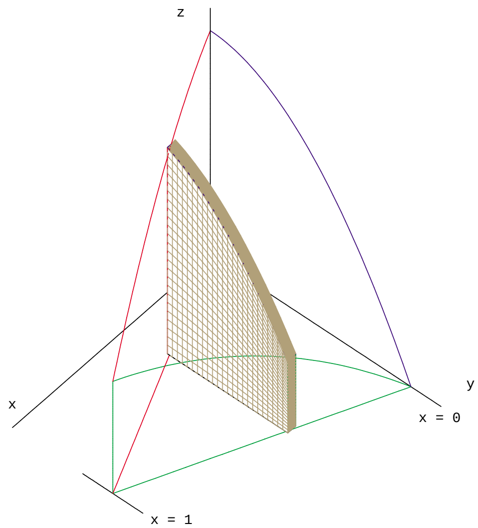
If you top this region with $z = 4 - x^2 - y^2$, then you get the figure below.



If we draw horizontal lines across the region we see that the lines will always start at $x = 0$, which is the y-axis, but they can end on two different curves, $y = x$ and $y = 2 - x$. This would mean that integrating first with respect to x, which changes from left to right, would violate the 'curve-to-curve' concept for our inside integral. As you see in the right-hand plot, every vertical line begins at $y = x$ and ends on $y = 2 - x$. Think of y, which increases by moving up, and you see that the inside integral can be with respect to y. Visualize the small rectangle as it slides up the longer one. Now, imagine that little rectangle as the base of a vertical column inside the solid. As the volume of each column is added up along the long rectangle we obtain the volume of a "slab" that is parallel to the yz-plane. Think of the first integration as determining the volumes of all those slabs. The second integration adds up the volumes of the slabs to get the volume of the solid. At what number does the 'first' slab occur? At what number does the 'last' slab occur? That is the "point-to-point" part of the process. This idea is illustrated in the plots that follow. First, a value is established for each column along the strip that is parallel to the y-axis. The diagram on the left shows the column in outline form and the one on the right shows the column as a solid. It is important here to realize that the values for each column are summed between two curves, $y = g(x)$ and $y = h(x)$, as the first step in the process.



occurs between two constants, or points. Here the points are $x=0$ and $x=1$. Visualize this as beginning with a slab in the yz-plane where $x=0$ and stacking slabs in the positive x-direction until you reach the value (point) $x=1$.



We are ready to start the set-up process. We know that we must integrate with respect to y first. We must complete this iterated integral in the format shown:

$$\int_{x=}^{x=}\int_{y=}^{y=} 4-x^2-y^2\,dy\,dx = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} 4-x^2-y^2\,dy\,dx$$

This means that the inner integral will be

$$\int_{y=x}^{y=2-x} 4-x^2-y^2\,dy$$

Think of a vertical line sliding from left to right over the region. The first x value that occurs is $x=0$ and the last is $x=1$. So our double integral becomes

$$\int_{x=0}^{x=1}\int_{y=x}^{y=2-x} 4-x^2-y^2\,dy\,dx$$

BEWARE! In Maple, we must be sure to put the inner integral limits first and the outer ones last.

```
> A:=Doubleint(4-x^2-y^2,y=x..(2-x),x=0..1);
```

$$A := \int_0^1\int_x^{2-x} 4-x^2-y^2\,dy\,dx$$

Incidentally, the value of this integral is $\frac{8}{3}$. Compare the syntax for the double integral to the equivalent non-parametric syntax we would have used to plot the top in the graphing section. The similarities are obvious.

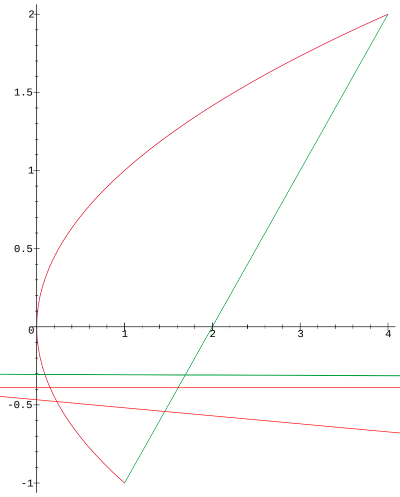
```
> A3:=plot3d(4-x^2-y^2,y=x..(2-x),x=0..1,color=blue):
```

Example: Sketch the solid bounded by $x=y^2,\ x=y+2,\ z=0,\ x+z=4$ and find its volume.

Eliminating x from the first two equations yields $y^2=y+2,\ 0=y^2-y-2=(y-2)(y+1)$, so the vertical parabolic surface meets the vertical plane at $y=-1$ and $y=2$. The horizontal plane $z=0$ will serve as a bottom and the oblique plane $x+z=4$ as a top. The domain in the plane is easy to plot parametrically. The output is on the left below.

```
> A1:=plot([y^2,y,y=-1..2],color=red):
> A2:=plot([y+2,y,y=-1..2],color=green):
```

```
> display(A1,A2);
```



C3M9

Double Integrals in Polar Coordinates

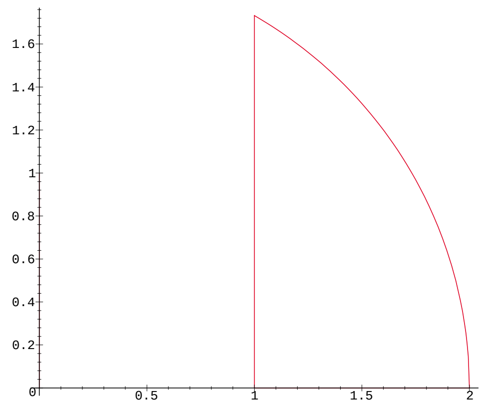
There are regions in the plane that are not easily used as domains of iterated integrals in rectangular coordinates. Sometimes switching to an integral in polar coordinates makes a difficult problem much easier. We will begin with a simple example and show how to make the transition. There are certain *Do's* and *Don't's* that we will point out. First, **ALWAYS** begin by writing the variables of integration in the limits of the **rectangular** integral, such as "$x =$" and "$y =$". Second, **ALWAYS** draw a sketch of the domain of the integral after the first step and think about how you would draw that same sketch using polar coordinates.

Example 1: Evaluate the integral $\int_1^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx$ using polar coordinates.

$$\text{First Step :} \quad \int_1^2 \int_0^{\sqrt{4-x^2}} y\,dy\,dx \quad \longrightarrow \quad \int_{x=1}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} y\,dy\,dx$$

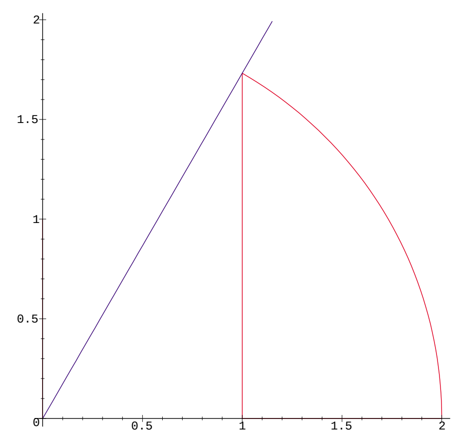
Let's use Maple to plot the domain. We will see that we have a portion of the circle of radius 2 in the first quadrant. A fourth plot is used to put the axes in their proper place. The output is to the right below.

```
> with(student):    with(plots):
> A:=plot([x,sqrt(4-x^2),x=1..2]):
> B:=plot([1,y,y=0..sqrt(3)]):
> C:=plot([x,0,x=1..2]):
> E:=plot([0,y,y=0..1]):
> display(A,B,C,E);
```



We are providing more details here than are really necessary, but maybe someone will benefit from it. Now we will plot the same region using `polarplot`. The output is below on the right. We must use the fact that a vertical line $x = a$ in polar coordinates is represented by $r = a\sec\theta$. If $a = 0$ then we use $\theta = \pi/2$.

```
> A1:=polarplot([2,t,t=0..Pi/3]):
> B1:=polarplot([sec(t),t,t=0..Pi/3]):
> C1:=polarplot([r,0,r=1..2]):
> angleline:=polarplot([r,Pi/3,r=0..2.2],color=blue):
> display(A1,B1,C1,E,angleline);
```



We included the plot `angleline` to show the last angle needed to plot the figure. Now we are ready to set up our integral. **NEVER** simply substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the limits of the rectangular integral. **ALWAYS** make those substitutions into the integrand $f(x,y)$ and multiply the result by r to form $f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$. Look at the figure and ask yourself, "How do I draw this figure in polar coordinates?" "Which functions of θ does r range between?" And "For which values does θ first touch the figure and last touch the figure?" The process looks like

$$\int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx \quad \longrightarrow \quad \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=u(\theta)}^{r=v(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

Think of a line emanating from the origin and extending out at about 30°. It will first touch the figure on the vertical line and exit the figure at the circle. And, this will remain true as you place that line along the x-axis and rotate it up to the point where the line and arc intersect. Using Maple to substitute into the integrand:

```
> f:=(x,y)->y;
```

$$f := (x,y) \longrightarrow y$$

```
> grand:=simplify(f(r*cos(t),r*sin(t)),symbolic);
```

$$grand := r\,\sin(t)$$

The polar integral becomes

$$\int_{\theta=0}^{\theta=\pi/3} \int_{r=\sec(\theta)}^{r=2} r\,\sin(\theta)\,r\,dr\,d\theta = \int_0^{\pi/3} \int_{\sec(\theta)}^{2} r^2\,\sin(\theta)\,dr\,d\theta$$

```
> answer1:=Doubleint(grand*r,r=sec(t)..2,t=0..Pi/3);
```

$$answer1 := \int_0^{\pi/3} \int_{\sec(t)}^{2} r^2\,\sin(t)\,dr\,dt$$

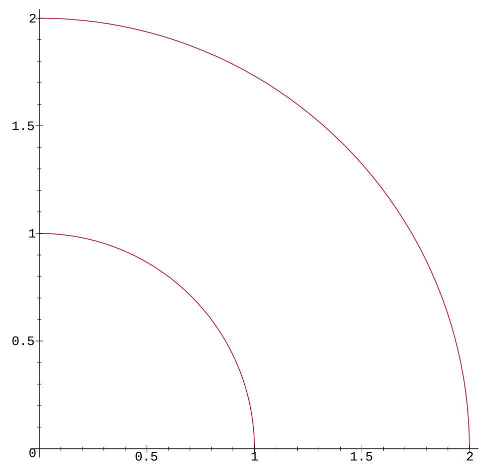
```
> answer2:=value(answer1);
```

$$answer2 := \frac{5}{6}$$

Example 2: Find the integral of $f(x,y) = e^{x^2+y^2}$ over the annular region in the first quadrant between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

It is easiest to graph this using `polarplot`.

```
> with(student):    with(plots):
> P1:=polarplot([2,t,t=0..Pi/2]):
> P2:=polarplot([1,t,t=0..Pi/2]):
> display(P1,P2);
```



It is easy to see that this integral cannot be done as one iterated integral in rectangular coordinates.

```
> f:=(x,y)->exp(x^2+y^2);
```

$$f := (x,y) \longrightarrow e^{(x^2+y^2)}$$

```
> grand:=simplify(f(r*cos(t),r*sin(t)),symbolic);
```

$$grand := e^{(r^2)}$$

```
> Polint:=Doubleint(grand*r,r=1..2,t=0..Pi/2);
```

$$Polint := \int_0^{1/2\,\pi} \int_1^2 e^{(r^2)} r\,dr\,dt$$

```
> Polint:=value(Polint);
```

$$Polint := \frac{1}{4}\pi e^4 - \frac{1}{4}\pi\,e$$

In addition to problems with the region, there is no known antiderivative for e^{x^2}. The presence of the extra r in the polar integrals makes such problems a straightforward substitution, $u = r^2$.

C3M9 Problems Use Maple and polar coordinates to evaluate the given integrals.

1. $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$

2. $\displaystyle\iint_S \frac{x^2}{x^2+y^2}\,dA$, S is the annular region between $x^2+y^2=1$ and $x^2+y^2=3$ with $y \geq 0$.

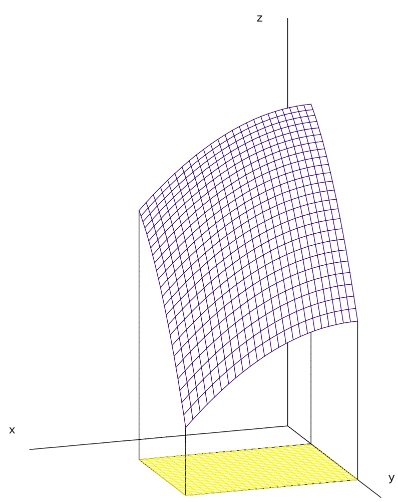
3. $\displaystyle\iint_R \sqrt{x^2+y^2}\,dA$, R is the triangle with vertices $(0,0)$, $(3,0)$, and $(3,3)$.

Challenge! $\displaystyle\int_{0}^{2}\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} x\,dx\,dy$

C3M10

Surface Area

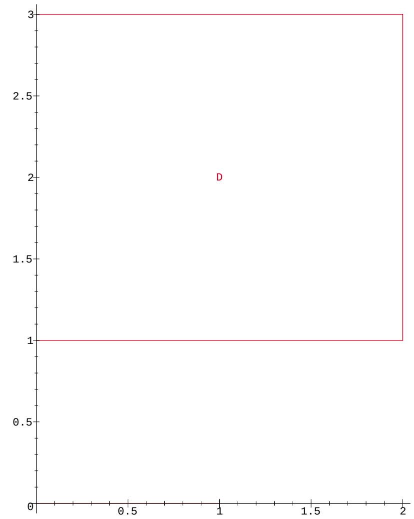
There are two ways to define a surface in 3-space. If we think of $z = f(x,y)$ with a domain D in the plane (2-space, or $\mathbb{R}^2$), then we visualize the surface as those points with coordinates $(x, y, f(x,y))$ which "lie over" the set D which is in the xy-plane, shown here in yellow. This is the first approach.



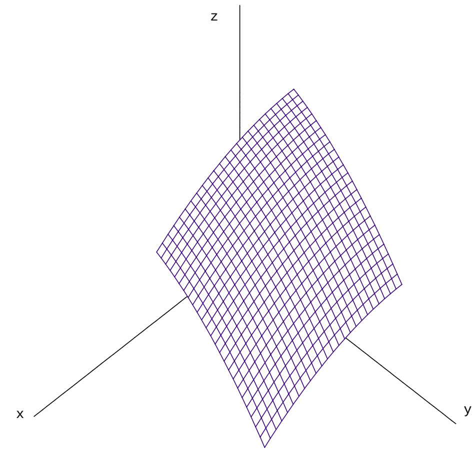
This is actually a special case of defining a surface parametrically. If D is a set in $\mathbb{R}^2$ and g is defined as

$$g\colon D \to \mathbb{R}^3 \qquad g(u,v) = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \\ g_3(u,v) \end{pmatrix}$$

then our surface is defined as the image $g(D) = S$.

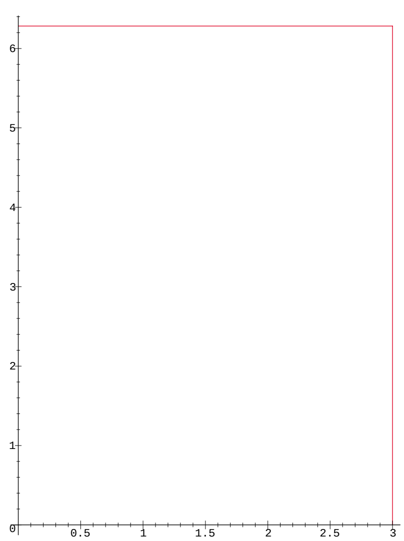


$\xrightarrow{g}$

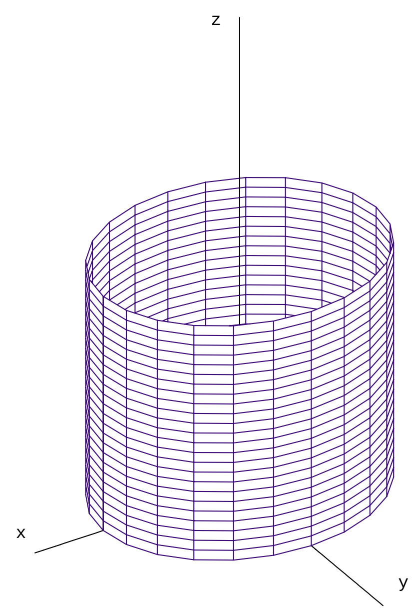


Example 1. Suppose $z = f(x, y)$ with domain D. Define $g(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$ where it is obvious that u and v play the role of x and y.

Example 2. Suppose that $D = \{(u, v) : \ 0 \le u \le 3, 0 \le v \le 2\pi\}$ and $g(u, v) = \begin{pmatrix} 2\cos(v) \\ 2\sin(v) \\ u \end{pmatrix}$. Maybe you would be more comfortable with $g(z, \theta) = \begin{pmatrix} 2\cos(\theta) \\ 2\sin(\theta) \\ z \end{pmatrix}$. As θ changes from 0 to 2π, the x and y coordinates go around a circle of radius 2. The variable z is the vertical coordinate and assumes all values between 0 and 3. We have parameterized the curved surface of a cylinder of radius 2 and height 3.



$\xrightarrow{g}$



Example 3. Consider

$$g(u, v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u \end{pmatrix} \qquad \begin{matrix} 0 \le u \le 2 \\ 0 \le v \le 2\pi \end{matrix}$$

or equivalently,

$$g(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix} \qquad \begin{matrix} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{matrix}$$

If r is held constant and θ varies, then you obtain a circle of radius r in the plane $z = r$. If θ is held constant at $\theta = 0$, and r varies, the result is a line in the vertical plane $y = 0$ and the slanted plane $z = x$. Now rotate that around the z-axis and you will understand how we have parameterized an inverted cone with base radius 2 and height 2. ($x^2 + y^2 = z^2, 0 \le z \le 2$). Look at the x and y components of g and you will see that a disk of radius 2 is parameterized. But as r increases, so does z, which yields a cone instead of a disk.

If we compute $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ then we obtain the vector valued functions

$$\frac{\partial g}{\partial u} = \begin{pmatrix} \frac{\partial g_1}{\partial u} \\ \frac{\partial g_2}{\partial u} \\ \frac{\partial g_3}{\partial u} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial v} = \begin{pmatrix} \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial v} \end{pmatrix}$$

which may be regarded as tangent vectors at a point on the surface. Visualize the parallelogram that these two tangent vectors would generate. How would we find the area of that parallelogram? Simple! Take their cross product and find its length! We have a name for $\dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v}$, it is the **fundamental cross product**. The parallelogram we generated may be seen loosely as the image of a unit square in the domain, D, generated by a translation of the unit vectors, $\overrightarrow{i}$ and $\overrightarrow{j}$. The length of the fundamental cross product is like a magnification factor for the square whose area is 1 unit. Multiplying by $du\,dv$ we obtain $d\sigma = \left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\| du\,dv$, which is regarded as an element of surface area. If we integrate $d\sigma$ over the domain, D, then we obtain the area of our surface S, $\sigma(S)$.

$$\sigma(S) = \iint_D d\sigma = \iint_D \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\,dv$$

Example 1, revisited. Let's compute the fundamental cross product for g.

$$g(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix} \qquad \frac{\partial g}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix} \qquad \text{and} \qquad \frac{\partial g}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$$

The cross product and $d\sigma$

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} = \begin{vmatrix} 1 & 0 & f_u \\ 0 & 1 & f_v \\ \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \end{vmatrix} = \langle -f_u, -f_v, 1 \rangle \qquad d\sigma = \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\,dv = \sqrt{(f_u)^2 + (f_v)^2 + 1}\,du\,dv$$

which explains the formula for surface area that you see when given $z = f(x,y)$ over D.

$$\sigma(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\,dx\,dy$$

Example 4. Find the surface area of that portion of the plane $2x+3y+4z = 36$ that lies over the rectangle $D = \{(x,y) : \ 0 \le x \le 3, 0 \le y \le 2\}$.

Solving for z, $z = 9 - x/2 - 3y/4$. So $\sqrt{(f_x)^2 + (f_y)^2 + 1} = \sqrt{(-1/2)^2 + (-3/4)^2 + 1} = \sqrt{29/16} = \sqrt{29}/4$. So,

$$\sigma(S) = \int_0^3 \int_0^2 \frac{\sqrt{29}}{4}\,dy\,dx = \frac{\sqrt{29}}{4} \cdot 2 \cdot 3 = \frac{3\sqrt{29}}{2}$$

A little thought here allows us to realize that because the length of the fundamental cross product is a constant, if we know the area $\sigma(D)$ of any domain D, the area of the portion of the given plane 'above' D is always just $\frac{\sqrt{29}}{4} \cdot \sigma(D)$. This explains why we referred earlier to the length of the fundamental cross product as a magnification factor.

Example 5 (Maple): Use Maple to find the area of the cone parameterized in Example 2. We will define g as a vector expression. Note the syntax when we take the partial derivatives of the vector expression. The command `map` allows Maple to differentiate on each component separately. It is best to wait until after the length of the fundamental cross product has been computed before trying to simplify. Then you are dealing with an expression rather than a vector expression, which is less likely to cooperate.

```
> restart:  with(student):  with(linalg):
> g:=vector([u*cos(v),u*sin(v),u]);
```

$$g := [u\cos(v), u\sin(v), u]$$

```
> gu:=map(diff,g,u);
```

$$gu := [cos(v), sin(v), 1]$$

```
> gv:=map(diff,g,v);
```

$$gv := [-u\sin(v), u\cos(v), 0]$$

```
> fcp:=crossprod(gu,gv);
```

$$fcp := [-4\cos(v), -u\sin(v), \cos(v)^2 u + \sin(v)^2 u]$$

```
> grand:=norm(fcp,2);
```

$$grand := \sqrt{|u\cos(v)|^2 + |u\sin(v)|^2 + |\cos(v)^2 u + \sin(v)^2 u|^2}$$

```
> grand:=simplify(grand,symbolic);
```

$$grand := \sqrt{2}\,u$$

```
> surfacearea:=Doubleint(grand,u=0..2,v=0..2*Pi);
```

$$surfacearea := \int_0^{2\pi}\int_0^2 \sqrt{2}\,u\,du\,dv$$

```
> surfacearea:=value(surfacearea);
```

$$surfacearea := 4\pi\sqrt{2}$$

Problems: Do these problems by pencil and paper.

1. Compute the surface area of the curved surface of the cylinder in Example 2 using the approach shown in these notes.

2. Find the area of the portion of the plane that contains the points $P(1,3,1)$, $Q(0,2,3)$, and $R(1,1,3)$ and is above the triangle with vertices $V_1(1,0,0)$, $V_2(2,1,0)$, $V_3(1,2,0)$.

3. Compute the surface area of the paraboloid $z = 9 - x^2 - y^2, z \geq 0$. Hint: parameterize this surface using polar coordinates. Then $z = 9 - r^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $D = \{(r,\theta) :\ 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

C3M10 Problems: Turn these in as **C3M10**

4. Use Maple to compute the surface area of the paraboloid $z = 9 - x^2 - y^2, z \geq 0$. Hint: parameterize this surface using polar coordinates. Then $z = 9 - r^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $D = \{(r,\theta) :\ 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

5. Use Maple to compute the area of the surface defined by $z = x^2 + y^2$ with domain $D = \{(x,y) :\ 1 \leq x^2 + y^2 \leq 4\}$.

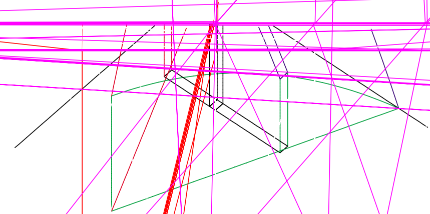
6. Use Maple to compute the area of the hemisphere $z = \sqrt{9 - x^2 - y^2}$ that is inside the vertical cylinder $x^2 + y^2 \leq 4$.

C3M11

The Evaluation of Triple Integrals by Iteration

The use of triple integrals involves two separate skills, the set up and the evaluation. In order to set up a triple integral we need to understand a simple phrase, **Surface-to-Surface, Curve-to-Curve, Point-to-Point**. The first or inner integral builds "columns" by adding "cubes" between two surfaces. The second integral adds up those "columns" that are bounded between two curves to obtain "slabs". The third integral adds up the "slabs" between two points to get a volume. This simplistic view of things pre-supposes that the function $f(x, y, z) = 1$ was the integrand. The reader should realize that if the integrand is a density function, such as $\delta(x, y, z)$, whose units are "pounds per cubic foot" or "grams per cubic centimeter", then the evaluation of the integral will yield the weight of the solid in pounds or the mass m in grams respectively. So triple integrals compute far more than just volume. But the visualization of computing the volume by adding up small "cubes" as a means of establishing the limits of integration is an essential part of understanding the process of integration.

In the figure that follows one sees a vertical column which may be regarded as being formed from solid cubes. If we are finding the volume of a solid, then one begins the iterative process by adding up the volume of cubes to obtain the volume of a column. The cubes range between two surfaces while two variables are held constant. Here we see change with respect to z as z ranges between $z = 0$ and $z = f(x, y)$, while x and y are held constant.



You saw the last two figures in **C3M8**, so they should look familiar.

There is a very important idea that we need to understand from the beginning. If the integrand is continuous over the region or solid, then the integrand has **absolutely nothing** to do with establishing the limits of integration of the triple integral. These limits are determined solely by the geometry of the solid involved. It is the relationship of the solid to the coordinate planes that either permits the integral to be set up as one integral or it forces two or more integrals to be used. There are three variables, so there are six orders of integration that are possible, including $dx\,dy\,dz$ and $dy\,dz\,dx$ and so forth. Having said that, the process of successive anti-differentiation may range from easy to impossible when different possible orders are considered. We begin with a picture of a solid figure. After the first integral we look at that solid so that our line of sight is parallel to the columns we just built. We now see a two-dimensional view of the solid with the ends of the columns appearing as squares. We have reduced a triple integral to a double integral, which we know how to evaluate.

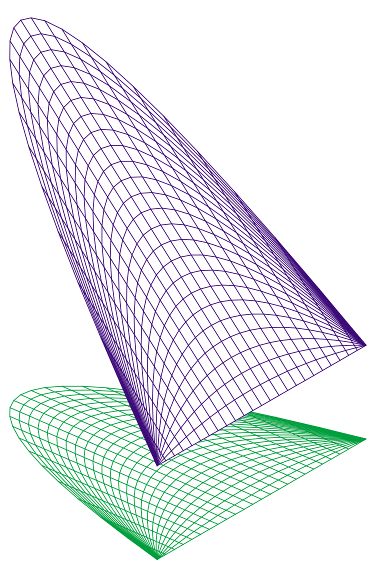
In a previous chapter we learned how to find partial derivatives. Essentially this involved differentiating with respect to one variable while the other variables were held constant. To evaluate a triple integral in iterated form we anti-differentiate with respect to one variable while the others are held constant, evaluate at the (variable) endpoints, and repeat this process with respect to a different variable. For all practical purposes we are "partially integrating" instead of "partially differentiating". If we begin by anti-differentiating with respect to z, then the integration takes place between two functions, say $z = g(x, y)$ and $z = h(x, y)$. It would look like

$$\int_{x=a}^{x=b} \int_{y=r(x)}^{y=s(x)} \left[F(x, y, z)\Big|_{z=g(x,y)}^{z=h(x,y)} \right] dy\,dx$$

after the first anti-differentiation. The curves $y = r(x)$ and $y = s(x)$ were seen in outline form as we looked down the z-axis after integrating with respect to z. It is important to note that after the substitution of $z = g(x, y)$ and $z = h(x, y)$ into $F(x, y, z)$, the variable z disappears and a double integral in terms of x and y remains.

Example 1 Suppose a solid is bounded by $y + z = 4$, $z = 0$, $y = x^2$, $y = 3$. Find the volume of the solid by using an iterated integral. The first or inner integral will have a roof $y + z = 4$ and a floor $z = 0$ as bounds. Because "floor" is a protected word in Maple, we will use "floor1" instead.

```
> with(student):with(plots):
> roof:=plot3d([x,y,4-y],x=-sqrt(3)..sqrt(3),y=x^2..3,color=blue):
> floor1:=plot3d([x,y,0], x=-sqrt(3)..sqrt(3),y=x^2..3,color=green):
> display(roof,floor1);
```

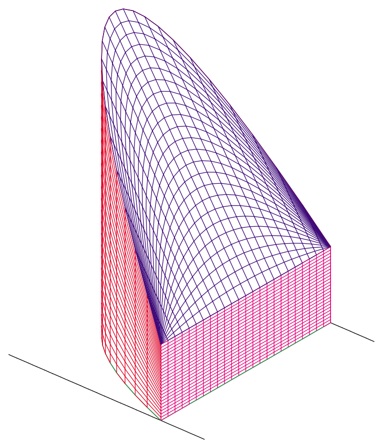


The integral for this part is

$$\int_{z=0}^{z=4-y} 1\,dz = 4 - y$$

Now we add the curved side $y = x^2$ and the vertical plane $y = 3$ as well as two lines which show where $x = -\sqrt{3}$ and $x = \sqrt{3}$..

```
> cside:=plot3d([x,x^2,z], x=-sqrt(3)..sqrt(3),z=0..4-x^2,color=red):
> vside:=plot3d([x,3,z], x=-sqrt(3)..sqrt(3),z=0..1,color=magenta):
> L1:=spacecurve([-sqrt(3),y,0],y=-.5..4,color=black):
> L2:=spacecurve([sqrt(3),y,0],y=-.5..4,color=black):
> display(roof,floor1,cside,vside,L1,L2);
```



Now we integrate between the curves $y = x^2$ and $y = 3$.

$$\int_{y=x^2}^{y=3} 4 - y\,dy = 4y - \frac{y^2}{2}\Big|_{y=x^2}^{y=3} = \left(12 - \frac{9}{2}\right) - \left(4x^2 - \frac{x^4}{2}\right) = \frac{15}{2} - 4x^2 + \frac{x^4}{2}$$

It remains to integrate this last result between the x values of $\pm\sqrt{3}$.

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{15}{2} - 4x^2 + \frac{x^4}{2}\,dx = \frac{44}{5}\sqrt{3}$$

The triple integral that accomplishes the same thing is

$$\int_{x=-\sqrt{3}}^{x=\sqrt{3}} \int_{y=x^2}^{y=3} \int_{z=0}^{z=4-y} 1\,dz\,dy\,dx$$

The Maple integral would be

```
> Q:=Tripleint(1,z=0..4-y,y=x^2..3,x=-sqrt(3)..sqrt(3));
```

$$Q := \int_{-\sqrt{3}}^{\sqrt{3}} \int_{y=x^2}^{y=3} \int_{z=0}^{z=4-y} 1\,dz\,dy\,dx$$

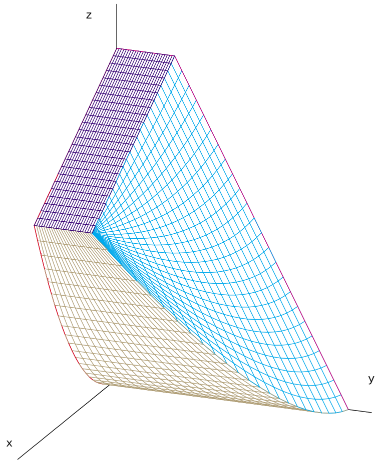
```
> value(Q);
```

$$\frac{44}{5}\sqrt{3}$$

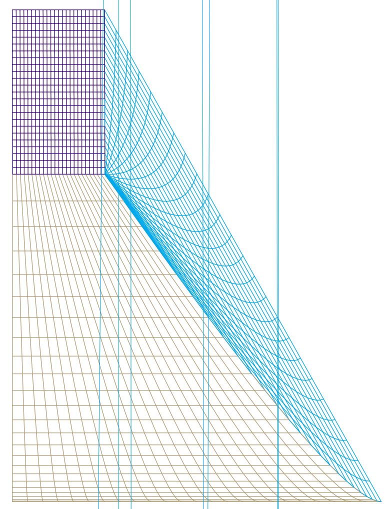
Compare the Maple syntax for the triple integral with that for drawing 'roof' and 'floor1'. The order of the limits in '`Tripleint`' determines the order of the variables to be integrated. Now let's consider an example where z is not the first variable to be integrated. In fact, let's really analyze how we must select the order of integration.

Example 2 Set up an iterated triple integral to find $\iiint_S x\,dV$ where S is bounded by $z = 2x^2$, $x+y+z = 4$, $x+z = 3$, $x = 0$, and $y = 0$.

We begin with a view of the solid to show the three surfaces which do not lie in a coordinate plane and the three views down each of the axes.

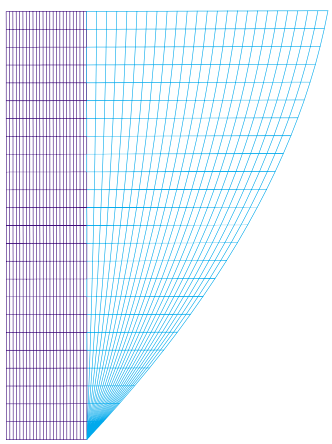


the solid



down the x-axis

From our view of the solid, we realize that we may not just assume that z is first. Looking down the x-axis we see three different surfaces. A light beam from behind the figure would always enter through the single surface where $x = 0$. But as it exits towards us, it could leave through *three* different surfaces, which denies the choice of integrating with respect to x first. Then we look down the z-axis from above the figure.



beam test allows the light to enter through the parabola and exit through the line when the light is below.

$$\text{Second step} \quad \int\int_{z=2x^2}^{z=3-x}\int_{y=0}^{y=4-x-z} x\,dy\,dz\,dx \quad \Longrightarrow \quad \int_{x=0}^{x=1}\int_{z=2x^2}^{z=3-x}\int_{y=0}^{y=4-x-z} x\,dy\,dz\,dx \quad \text{Third step}$$

Now consider the Maple syntax for this triple integral and the graphics which produced the solid as seen above when displayed, except for the coordinate axes.

```
> with(student):    with(plots):
> A2:=Tripleint(x, y=0..4-x-z, z=2*x^2..3-x, x=0..1);
```

$$A2 := \int_0^1\int_{2x^2}^{3-x}\int_0^{4-x-z} x\,dy\,dz\,dx$$

```
> value(A2);
```

$$\frac{51}{40}$$

```
> bottom:=plot3d([x,y,2*x^2],x=0..1,y=0..4-x-2*x^2,color=blue):
> top:=plot3d([x,y,3-x],x=0..1,y=0..1,color=red):
> slant:=plot3d([x,4-x-z,z],x=0..1,z=2*x^2..3-x,color=cyan):
> left:=plot3d([x,0,z],x=0..1,z=2*x^2..3-x,color=magenta):
> back:=plot3d([0,y,z],z=0..3,y=0..4-z,color=green):
> display(bottom,top,slant,left,back);
```

C3M11 Problems

Evaluate the triple integrals in 1, 2, and 3 by pencil and paper and by Maple to check your answers. Use Maple to plot the solid figure which is the domain of the integral. Hint: Insert $x =, y =, z =$ appropriately in the limits of the integrals before you begin, to help find equations for the surfaces and curves. Ignore the integrand when sketching the solid.

1. $A = \int_{-1}^{2}\int_0^{\pi}\int_1^4 yz\cos(xy)\,dz\,dx\,dy$
2. $B = \int_0^2\int_0^{3-x}\int_0^{6-x-y} x\,dz\,dy\,dx$
3. $C = \int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-2y} x\,dz\,dy\,dx$

In the remaining problems, use Maple to sketch the solid and to find the volume by evaluating the triple integral .

4. The solid is bounded by $x - y + z = 2,\ x = z^2,\ y = 0,\ x = 0$.
5. The solid is bounded by $z = 2 - x^2,\ x = z,\ x + y + z = 3,\ x = 0,\ y = 0$.
6. The solid is bounded by $z - y = 2,\ y = 2,\ y + z = 2,\ x - y^2 = 2,\ x = 0$.

C3M12a

Evaluating Integrals using Cylindrical Coordinates

Let's begin with an integral in rectangular coordinates, plot the solid which is the domain of the integral, plot the same solid using cylindrical coordinates, and then transform the integral into one using cylindrical coordinates. A simple first step is to include the variable of integration in the limits of the integral, which allows one to see the appropriate equations that define the surfaces. Immediately below the integral we will list the necessary packages and the Maple command that would produce this integral.

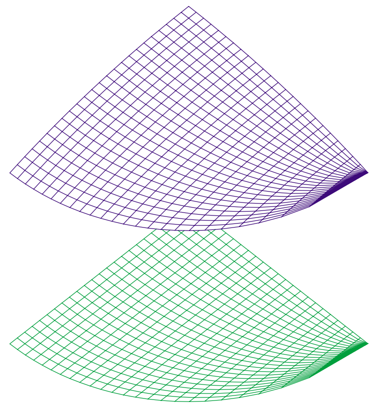
Example: $$\int_0^1\int_0^{\sqrt{1-y^2}}\int_0^{\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=0}^{z=\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy$$

```
> with(student):with(plots):
> A:=Tripleint(x*z,z=0..sqrt(4-x^2-y^2),x=0..sqrt(1-y^2),y=0..1);
> valA:=value(A);
```

$$valA := \frac{17}{30}$$

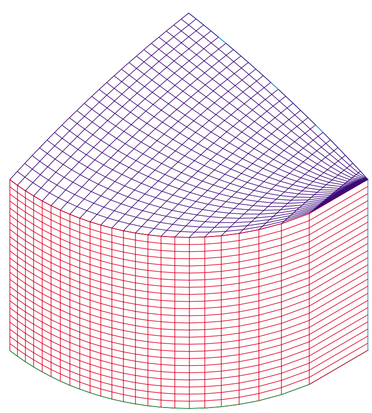
We will use `plot3d` in parametric form, `[x,y,z]`, with one of the variables eliminated so that just two are allowed to vary and that variable does not appear. The inner integral varies between two surfaces which, for intuitive reasons, we will designate as $floor1$ and $roof1$. Pay close attention to the similarities between the syntax for the triple integral and for these plots.

```
> floor1:=plot3d([x,y,0],x=0..sqrt(1-y^2),y=0..1,color=green):
> roof1:=plot3d([x,y,sqrt(4-x^2-y^2)],x=0..sqrt(1-y^2),y=0..1,color=blue):
> display(floor1,roof1);
```



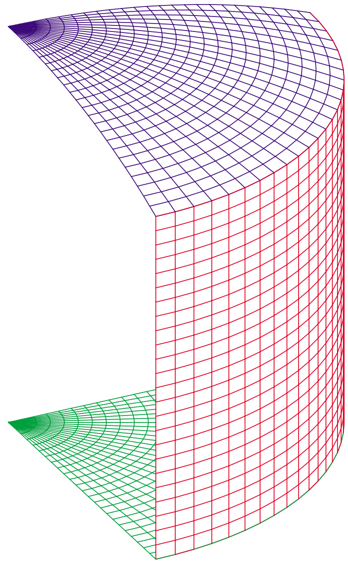
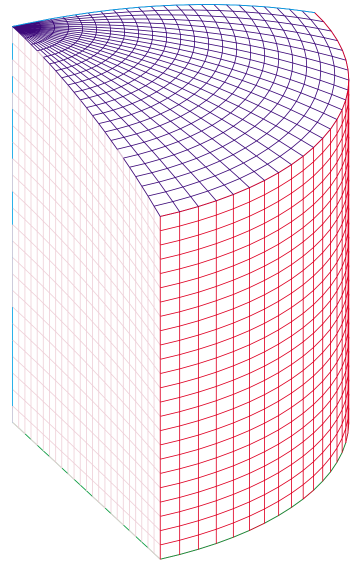
Now let's add the cylindrical wall, $cwall1$, and the two flat vertical walls.

```
> cwall1:=plot3d([sqrt(1-y^2)],y,z],z=0..sqrt(3),y=0..1,color=red):
> vwall1:=plot3d([0,y,z],z=0..sqrt(4-y^2),y=0..1,color=cyan):
> vwall2:=plot3d([x,0,z],z=0..sqrt(4-x^2),x=0..1,color=plum):
> display(floor1,roof1,cwall1,vwall1,vwall2);
```



We must see how to sketch this same solid in cylindrical coordinates. We know at this point that the solid is a quarter of a circular cylinder, cut off on the bottom by a horizontal plane ($z = 0$) and on the top by a sphere ($x^2 + y^2 + z^2 = 4$). All of this takes place in the first octant which is where $x \geq 0$ and $y \geq 0$. You will find that this solid was discussed in the section on plotting using `cylinderplot`. We will use `cylinderplot`, which is a 3d-plotter using coordinates $[r, \theta, z]$ in parametric form. It is easy to see that we have used t instead of θ here. The figure below has been rotated to provide a better viewing angle. In cylindrical coordinates we have the following with its output on the left below.

```
> roof2:=cylinderplot([r,t,sqrt(4-r^2)],r=0..1,t=0..Pi/2,color=blue):
> floor2:=cylinderplot([r,t,0],r=0..1,t=0..Pi/2,orientation=[-26,69],color=green):
> cwall2:=cylinderplot([1,t,z],t=0..Pi/2,z=0..sqrt(3),color=red):
> display(floor2,roof2,cwall2);
```

Adding two more walls gives the figure on the right above as the output of the next commands.

```
> vwall3:=cylinderplot([r,Pi/2,z],z=0..sqrt(4-r^2),r=0..1,color=plum):
> vwall4:=cylinderplot([r,0,z],z=0..sqrt(4-r^2),r=0..1,color=cyan):
> display(floor2,roof2,cwall2,vwall3,vwall4);
```

We are almost ready to set up our triple integral but first must determine the integrand. This is the one part of the process that is dealt with as a simple direct substitution. In cylindrical coordinates we have

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

Our integrand in rectangular coordinates is $f(x,y,z) = xz$, so in Maple

```
> f:=(x,y,z)->x*z;
```

$$f := (x, y, z) \to xz$$

```
> F:=simplify(f(r*cos(t),r*sin(t),z));
```

$$F := r\,\cos(t)\,z$$

IMPORTANT: Note the use of **F*r** for the integrand in our cylindical coordinate integral! Also, note that **f** is a function, while **F** is an expression, as is **F*r**.

```
> B:=Tripleint(F*r,z=0..sqrt(4-r^2),r=0..1,t=0..Pi/2);
```

$$B := \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{4-r^2}} r^2\,\cos(t)\,z\,dz\,dr\,dt$$

```
> Bval:=value(B);
```

$$Bval := \frac{17}{30}$$

With the limits displayed fully, we have

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=0}^{z=\sqrt{4-x^2-y^2}} xz\,dz\,dx\,dy = \int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} r^2\cos(\theta)z\,dz\,dr\,d\theta\,.$$

C3M12a Problems

Use `cylinderplot` to plot the solid which represents the domain of the integral. Use Maple to find the value of an equivalent integral in cylindrical coordinates.

1. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

2. $\displaystyle\int_{0}^{2}\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}}\int_0^{x^2+y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

Evaluating Integrals Using Spherical Coordinates

We will begin with plotting using '`sphereplot`' and then address the integrals. In parametric form `sphereplot` uses $[\rho, \theta, \phi]$, where one of the three is in terms of the other two or is a constant. Remember that ϕ is measured down from the "north pole" or the usual z-axis. To understand the relationships between the coordinate systems it is good to start with the r of polar or cylindrical coordinates and recall that

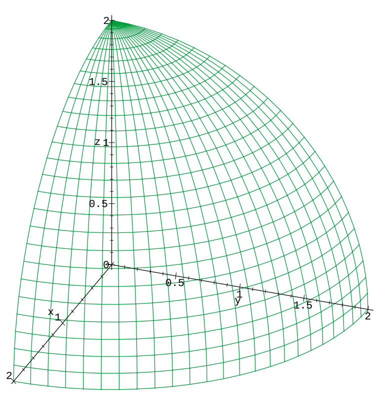
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Building on this, use the fact that $r = \rho\sin\phi$ to obtain for $[\rho, \theta, \phi]$

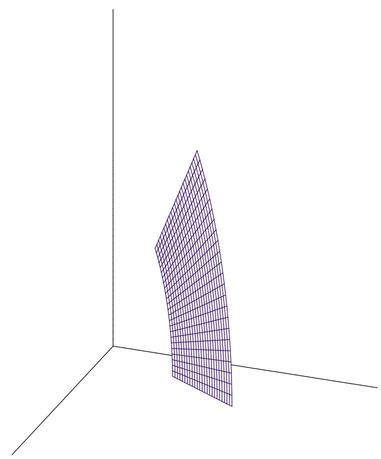
$$\begin{aligned} x &= (r)\cos\theta \\ y &= (r)\sin\theta \\ z &= \rho\cos\phi \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

It helps if we understand what it means for each of the variables to be held constant while the other two vary. In the first of three simple plots we hold ρ constant. This should produce some portion of a sphere. In order to have the axes appear at the origin we have included a graph in white which we do not list here. It is not necessary to use the Greek names of the variables, but for demonstration purposes it is easiest. The output is on the left.

```
> sphereplot([2,theta,phi],theta=0..Pi/2,phi=0..Pi/2,color=green);
```



ρ is constant



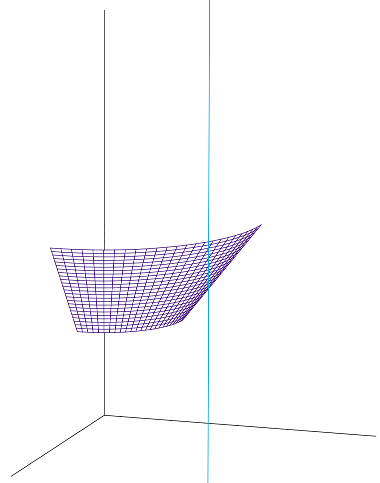
θ is constant

With the output above on the right, hold θ constant. The surface should be in the vertical plane $\theta = \pi/3$.

```
> sphereplot([rho,Pi/3,phi],rho=1..2,phi=Pi/4..Pi/2,color=blue);
```

Now hold ϕ constant. This restricts the surface to a portion of some cone.

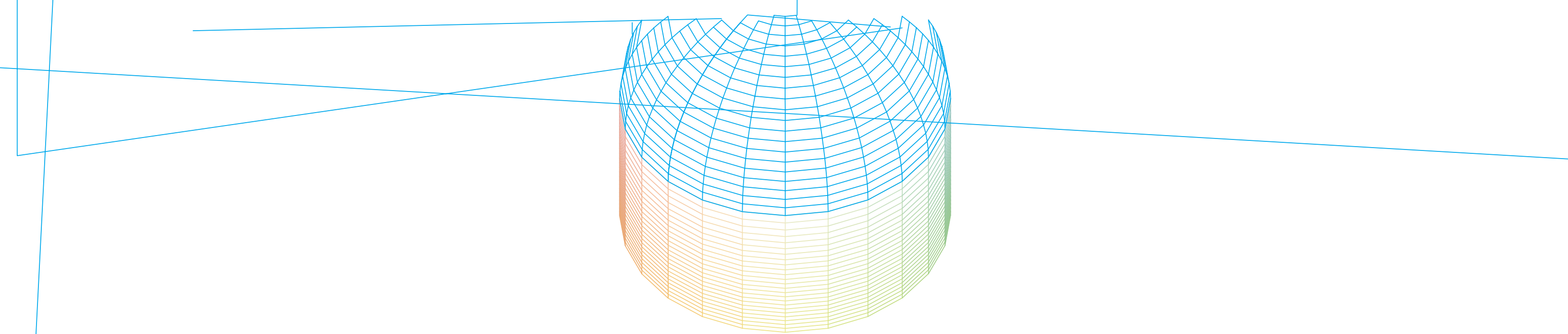
```
> sphereplot([rho,theta,Pi/3],rho=1..2,theta=0..Pi/2,color=blue);
```

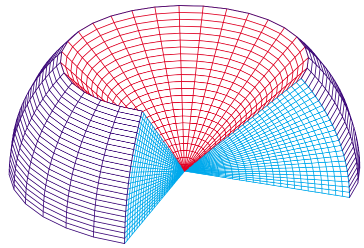


ϕ is constant

Recall that in polar coordinates $r = a\sec\theta$ produced a vertical line, $x = a$. And, $r = a\csc\theta$ produced a horizontal line, $y = a$. The equation $r = a\cos\theta$ yielded a circle, but here, the angle ϕ is measured from the z-axis, so $\rho = a\cos\phi$ would produce a circle on the z-axis for *every* θ. This results in a sphere tangent to the xy-plane. Let's combine that with an equation of the form $\rho = a\csc\phi$, which is a vertical cylinder of radius a.

```
> A:=sphereplot([4*cos(phi),theta,phi],theta=0..2*Pi,phi=0..Pi/4,color=cyan):
> B:=sphereplot([2*csc(phi),theta,phi],theta=0..2*Pi,phi=Pi/4..Pi/2):
> display(A,B);
```





```
> V1:=Tripleint(rho^2*sin(phi),rho=0..3,phi=Pi/4..Pi/2,theta=Pi/2..2*Pi);
```

$$V1 := \int_{\pi/2}^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

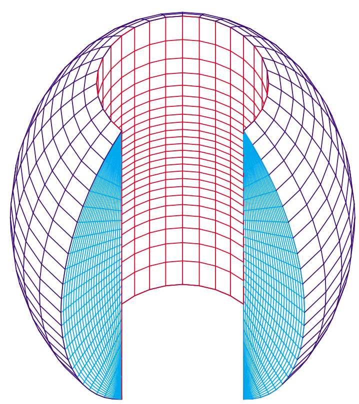
```
> valV1:=value(V1);
```

$$valV1 := \frac{27}{4}\pi\sqrt{2}$$

Example 2 Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4$ that is **outside** the cylinder $x^2 + y^2 = 1$.

A three-quarter view of this solid results from:

```
> sph2:=sphereplot([2,theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6,color=blue):
> cyl2:=sphereplot([csc(phi),theta,phi],theta=Pi/2..2*Pi,phi=Pi/6..5*Pi/6,color=red):
> side3:=sphereplot([rho,0,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6,color=cyan):
> side4:=sphereplot([rho,Pi/2,phi],rho=csc(phi)..2,phi=Pi/6..5*Pi/6,color=cyan):
> display(sph2,cyl2,side3,side4);
```



Integrate first with respect to ρ and imagine generating rays emanating from the origin which begin as they reach the cylinder and end as they exit through the sphere. Then add up those rays in vertical planes that contain the z-axis to form little "orange slices" by integrating with respect to ϕ from $\pi/6$ down to $5\pi/6$. End by integrating with respect to θ from the "first slice" at $\theta = 0$ and around to $\theta = 2\pi$.

$$\int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{\csc\phi}^{2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta = 4\pi\sqrt{3}$$

C3M12b Problems In problems 1 and 2, plot the solid and find its volume using spherical coordinates and Maple.

1. Q is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$. A horizontal plane in spherical coordinates has the form $\rho = a \sec\phi$.

2. R is the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 6z$.

3. Evaluate the integral by changing to spherical coordinates:

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$

C3M13

Potential Functions

Before we learn how to evaluate line integrals it is helpful to get some of the terminology straight and learn how different mathematical objects relate to each other. By this time we have established that a function $\overrightarrow{F} : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a *vector-valued function* or a *vector field*. For example, $\overrightarrow{F}(x, y) = \langle x - y, 2x + y^2\rangle$ is a simple example of such an object. However, the best and most important example we have is when we take the gradient of a real-valued function of several variables. For example,

$$f(x, y) = e^{2x}\sin 3y \qquad \text{yields the gradient} \qquad \nabla f = \langle 2e^{2x}\sin 3y,\ 3e^{2x}\cos 3y\rangle$$

Definition: A vector field $\overrightarrow{F}$ is *conservative* if there is a real-valued function g so the $\nabla g = \overrightarrow{F}$. In this case, we say that g is a *potential function* for $\overrightarrow{F}$.

Question: Given a vector field, is there a test to determine whether the vector field has a potential function? Or the same question, phrased differently - How do we know when a vector field is conservative?

The answer lies in this theorem. A more expansive version will be stated in a later section.

THEOREM. Suppose $\overrightarrow{F}$ is defined on an open simply connected region D. If the component functions of $\overrightarrow{F}$ have continuous partial derivatives and the curl of $\overrightarrow{F}$ is zero, then $\overrightarrow{F}$ is conservative.

(a) (2-dim) If (x_0, y_0) is in D, then we may define a potential function, g, of $\overrightarrow{F} = \langle M, N\rangle$ by

$$g(x, y) = \int_{x_0}^{x} M(x, y_0)\, dx + \int_{y_0}^{y} N(x, y)\, dy$$

(b) If (x_0, y_0, z_0) is in D and $\overrightarrow{F} = \langle M, N, P\rangle$, we may define g by

$$g(x, y, z) = \int_{x_0}^{x} M(x, y_0, z_0)\, dx + \int_{y_0}^{y} N(x, y, z_0)\, dy + \int_{z_0}^{z} P(x, y, z)\, dz$$

Not only was our question answered, but we are given a method for finding the elusive potential function. There are other methods, but this one is a little more systematic. Before we examine parts (a) and (b) above, let's examine what it means for the *curl of $\overrightarrow{F}$ to be zero.* Because we will discuss divergence and curl in a later section, let's just cut to the chase.

(A) If given $\overrightarrow{F}(x, y) = \langle M(x, y), N(x, y)\rangle$, then $\overrightarrow{F}$ is conservative if, and only if, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(B) If given $\overrightarrow{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$, then $\overrightarrow{F}$ is conservative if, and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

.

Part (A) is the two-dimensional statement that the curl is 0, while part (B) is the three-dimensional statement.

Example 1 If $\overrightarrow{F}(x, y) = \langle 2x\cos y, -x^2\sin y\rangle$, then $M(x, y) = 2x\cos y$ and $N(x, y) = -x^2\sin y$ so that $\dfrac{\partial N}{\partial x} = -2x\sin y = \dfrac{\partial M}{\partial y}$.

Now we turn to the main objective of this section, specifically, to find the potential function for a conservative vector field. Looking at part (a) of the theorem above we see that $g(x, y)$ is found by evaluating an integral. We must choose a point (x_0, y_0), and it helps to be clever about it, and then integrate to an arbitrary point (x, y). The first integrand is $M(x, y_0)$, which is a pure function of x. Students are reluctant to substitute y_0 for y, but that is exactly what must happen. The second integrand is $N(x, y)$, where we regard x as a constant. In the case of N, the student must be careful to **NOT** substitute either x_0 or y_0 into $N(x, y)$. When the objective is to find $g(x, y)$ so that $\nabla g = \overrightarrow{F}$, then $g(x, y) + C$ will also work if g

is a solution. The process of finding a potential function is important. After we have learned about line integrals we will do some problems involving conservative vector fields and we will need to find a potential function to make some problems much simpler. Besides, the final exam always has a problem on it requiring the student to find a potential function.

Example 2 Suppose $\overrightarrow{F}(x,y) = \left\langle 3x^2\cos(2y) + \frac{y}{x},\, -2x^3\sin(2y) + \ln x \right\rangle$

(A) Find the domain D: Because of $\ln x$ we must assume that $x > 0$. Thus, $D = \{(x,y) : x > 0\}$.

(B) Prove that $\overrightarrow{F}$ is conservative.

This is done by showing that $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

Note that $\dfrac{\partial N}{\partial x} = -6x^2\sin(2y) + \dfrac{1}{x} = \dfrac{\partial M}{\partial y}$

(C) Find a potential function $g(x,y)$ for $\overrightarrow{F}$.

It is important to make a clever choice for (x_0, y_0). Many times the right selection of y_0 eliminates some portion of the first integral that we will use to find g.

Select $(x_0, y_0) = (1, 0)$. Then $M(x,0) = 3x^2\cos(0) + 0/x = 3x^2$ and

$$\begin{aligned}
g(x,y) &= \int_{x_0}^{x} M(x,y_0)\,dx \;+\; \int_{y_0}^{y} N(x,y)\,dy \\
&= \int_1^x 3x^2\,dx + \int_0^y -2x^3\sin(2y) + \ln x\,dy \\
&= x^3\Big|_1^x + x^3\cos(2y) + y\ln x\Big|_0^y \\
&= x^3 - 1 + x^3\cos(2y) + y\ln x - x^3 \\
g(x,y) &= x^3\cos(2y) + y\ln x
\end{aligned}$$

We dropped the -1 from g because the addition of a constant does not affect ∇g.

Example 2, with Maple We will use Maple to solve the problem again. Recall that

$$\overrightarrow{F}(x,y) = \left\langle 3x^2\cos(2y) + \frac{y}{x},\, -2x^3\sin(2y) + \ln x \right\rangle$$

```
> restart:    with(linalg):
> F:=(x,y)->[3*x^2*cos(2*y)+y/x,-2*x^3*sin(2*y)+ln(x)];
```

$$F := (x,y) \to \left[3x^2\cos(2y) + \frac{y}{x},\ -2x^3\sin(2y) + \ln(x)\right]$$

```
> F1y:=diff(F(x,y)[1],y);
```

$$F1y := -6x^2\sin(2y) + \frac{1}{x}$$

```
> F2x:=diff(F(x,y)[2],x);
```

$$F2x := -6x^2\sin(2y) + \frac{1}{x}$$

This allows us to conclude that $\overrightarrow{F}(x,y)$ is conservative. We choose $(x_0, y_0) = (1, 0)$.

```
> x0:=1;  y0:=0;
```

$$x0 := 1$$
$$y0 := 0$$

Let u be the dummy variable instead of x in the first integral and let v be the dummy variable instead of y in the second integral to establish g.

```
> grand1:=F(u,y0)[1];
```

$$grand1 := 3u^2$$

```
> grand2:=F(x,v)[2];
```

$$grand2 := -2x^3\sin(2v) + \ln(x)$$

```
> fng:=Int(grand1,u=x0..x)+Int(grand2,v=y0..y);
```

$$fng := \int_1^x 3u^2\,du + \int_0^y -2x^3\sin(2v) + \ln(x)\,dv$$

> `fng:=value(fng);`

$$fng := -x^3 - 1 + 2x^3\cos(y)^2 + \ln(x)\,y$$

> `g:=unapply(fng,(x,y));`

$$g := (x,y) \to -x^3 - 1 + 2x^3\cos(y)^2 + \ln(x)\,y$$

It looks as if Maple has applied the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ so that the function g in the earlier work on Example 2 appears different from that obtained here.

C3M13 problems You may do these by pencil and paper or with Maple. Verify that the given functions are conservative and find a potential function for each.

1. $\overrightarrow{F}(x,y) = \langle 3x^2 + 2x + y^2,\ 2xy + y^3\rangle$
2. $\overrightarrow{F}(x,y) = \langle \sin y + \dfrac{y}{x},\ x\cos y + \ln x\rangle$
3. $\overrightarrow{F}(x,y) = \langle e^x\cos(y^2) + \dfrac{y}{x^2},\ -2y\,e^x\sin(y^2) - \dfrac{1}{x}\rangle$
4. $\overrightarrow{F}(x,y,z) = \langle yz\,e^{xy},\ xz\,e^{xy} - \dfrac{1}{z},\ e^{xy} + \dfrac{y}{z^2}\rangle$

C3M14

Notes on Line Integrals

I. Some Basic Parameterizations.

The successful completion of problems involving line integrals often depends on one's ability to set up the vector functions, or spacecurves, to parameterize the paths involved. There are several basic types that we will need:

A. **Straight line joining two given points**. Suppose we wish to traverse the line segment from $P(2,-1,3)$ to $Q(-3,5,-1)$. In generic form we have

$$\overrightarrow{\alpha}(t) = (1-t)P + tQ \qquad 0 \le t \le 1$$

and specifically, for $0 \le t \le 1$

$$\begin{aligned}\overrightarrow{\alpha}(t) &= (1-t)\langle 2,-1,3\rangle + t\langle -3,5,-1\rangle\\ &= \langle 2-2t,\ -1+t,\ 3-3t\rangle + \langle -3t,\ 5t,\ -t\rangle\\ &= \langle 2-5t,\ -1+6t,\ 3-4t\rangle\end{aligned}$$

Note that $\overrightarrow{\alpha}(0) = P$ and $\overrightarrow{\alpha}(1) = Q$.

B. **Circle of radius a at origin traversed counterclockwise**. Use your polar coordinate experience and define

$$\overrightarrow{\alpha}(t) = \langle a\cos t,\ a\sin t\rangle \qquad 0 \le t \le 2\pi$$

And we may translate this circle to a new center (h,k) by adding $\langle h,k\rangle$.

$$\overrightarrow{\alpha}(t) = \langle h + a\cos t,\ k + a\sin t\rangle \qquad 0 \le t \le 2\pi.$$

C. **Ellipse traversed counterclockwise**. If the equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then use

$$\overrightarrow{\alpha}(t) = \langle a\cos t,\ b\sin t\rangle \qquad 0 \le t \le 2\pi.$$

D. **Elbow path**. These are handled in two parts. One begins each leg by identifying which variable does not change. For example, join $(2,1) \to (2,5) \to (-4,5)$. The first leg has $x = 2$ and one inserts that fact first. Then the other coordinate is handled naturally.

$$\overrightarrow{\alpha}_1(t) = \langle 2,\quad\rangle = \langle 2,t\rangle \qquad 1 \le t \le 5$$

The second leg is given by $\overrightarrow{\alpha}_2(t) = \langle\quad, 5\rangle = \langle 2-6t, 5\rangle \qquad 0 \le t \le 1.$

II. Some remarks concerning line integrals.

To begin our discussion, let's assume that a path C has been parameterized by a continuous function $\overrightarrow{\alpha} : [a, b] \to \mathbb{R}^2$, i.e. $\overrightarrow{\alpha}(t) = \langle x(t), y(t) \rangle$ on $[a, b]$. Also assume $\overrightarrow{F} = \langle M, N \rangle$ is a vector-valued function defined on a region that contains C. So $\overrightarrow{F}(x, y) = \langle M(x, y), N(x, y) \rangle$. If we let $\overrightarrow{r} = \langle x,\, y, \rangle$ then $d\overrightarrow{r} = \langle dx,\, dy \rangle$. This is just another way of writing $\overrightarrow{\alpha}\,'(t) = \langle x\,'(t), y\,'(t) \rangle$.

The line integral of $\overrightarrow{F}$ over the path C is defined by

$$\int_C M\,dx \,+\, N\,dy = \int_C \langle M, N \rangle \cdot \langle dx, dy \rangle = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_a^b \overrightarrow{F}(\overrightarrow{\alpha}(t)) \cdot \overrightarrow{\alpha}\,'(t)\,dt$$

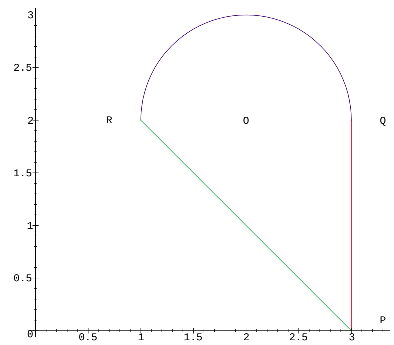
If $\overrightarrow{T}(t) = \overrightarrow{\alpha}\,'(t)/\|\overrightarrow{\alpha}\,'(t)\|$ and $\frac{ds}{dt} = \|\overrightarrow{\alpha}\,'(t)\|$ so that $ds = \|\overrightarrow{\alpha}\,'(t)\|\,dt$, then

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T}\,ds = \int_a^b \overrightarrow{F}(\overrightarrow{\alpha}(t)) \cdot \frac{\overrightarrow{\alpha}\,'(t)}{\|\overrightarrow{\alpha}\,'(t)\|}\,\|\overrightarrow{\alpha}\,'(t)\|\,dt = \int_a^b \overrightarrow{F}(\overrightarrow{\alpha}(t)) \cdot \overrightarrow{\alpha}\,'(t)\,dt = \int_C \overrightarrow{F} \cdot d\overrightarrow{r}$$

We mention this just to remind you that there are several notations that refer to the same mathematical entity. But the form we will use most often to evaluate line integrals is

$$\int_a^b \overrightarrow{F}(\overrightarrow{\alpha}(t)) \cdot \overrightarrow{\alpha}\,'(t)\,dt$$

Example 1 Suppose $\overrightarrow{F}(x, y) = \left\langle \dfrac{-y}{\sqrt{9x^2 + 4y^2}}, \dfrac{x}{\sqrt{9x^2 + 4y^2}} \right\rangle$ and C is the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ traversed counterclockwise. Evaluate $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$.



Because the path C has three distinct parts, we have three line integrals whose values are added together to find the solution.

Path C_1: Line segment from $P(3,0)$ to $Q(3,2)$. Use $\vec{\alpha}_1(t) = \langle 3, 2t\rangle$ for $0 \le t \le 1$.

```
> restart:    with(linalg):    with(student):
> F:=(x,y)->[-y,x];
```

$$F := (x, y) \to [-y, x]$$

```
> alpha1:=[3,2*t];
```

$$\alpha 1 := [3, 2t]$$

```
> Falpha1:=F(op(alpha1));
```

$$Falpha1 := [-2t, 3]$$

```
> alpha1prime:=map(diff,alpha1,t);
```

$$alpha1prime := [0, 2]$$

```
> grand1:=innerprod(Falpha1,alpha1prime);
```

$$grand1 := 6$$

```
> Lint1:=Int(grand1,t=0..1);
```

$$Lint1 := \int_0^1 6\, dt$$

```
> L1:=value(Lint1);
```

$$L1 := 6$$

Path C_2: Semicircular arc, radius 1, center at $O(2,2)$. Use $\vec{\alpha}_2(t) = \langle \cos(t)+2, \sin(t)+2\rangle$ for $0 \le t \le \pi$.

```
> alpha2:=[cos(t)+2,sin(t)+2];
```

$$\alpha 2 := [\cos(t) + 2, \sin(t) + 2]$$

```
> alpha2prime:=map(diff,alpha2,t);
```

$$alpha2prime := [-\sin(t), \cos(t)]$$

```
> Falpha2:=F(op(alpha2));
```

$$Falpha2 := [-\sin(t) - 2, \cos(t) + 2]$$

```
> grand2:=innerprod(Falpha2,alpha2prime);
```

$$grand2 := \sin(t)^2 + 2\sin(t) + \cos(t)^2 + 2\cos(t)$$

```
> grand2:=simplify(grand2,symbolic);
```

$$grand2 := 2\sin(t) + 2\cos(t) + 1$$

```
> Lint2:=Int(grand2,t=0..Pi);
```

$$Lint2 := \int_0^{\pi} 2\sin(t) + 2\cos(t) + 1\, dt$$

```
> L2:=value(Lint2);
```

$$L2 := 4 + \pi$$

Path C_3: Line segment from $R(1,2)$ to $P(3,0)$. Use $\vec{\alpha}_3(t) = \langle t, 3-t\rangle$ for $1 \le t \le 3$.

```
> alpha3:=[t,3-t];
```

$$\alpha 3 := [t, 3 - t]$$

```
> alpha3prime:=map(diff,alpha3,t);
```

$$alpha3prime := [1, -1]$$

```
> Falpha3:=F(op(alpha3));
```

$$Falpha3 := [-3 + t, t]$$

```
> grand3:=innerprod(Falpha3,alpha3prime);
```

$$grand3 := -3$$

```
> Lint3:=Int(grand3,t=1..3);
```

$$Lint3 := \int_1^3 -3\,dt$$

```
> L3:=value(Lint3);
```

$$L3 := -6$$

Combining our answers we get

```
> Lineint:=L1+L2+L3;
```

$$Lineint := 4 + \pi$$

Because C was a closed path, we will revisit Example 2 when we discuss Green's Theorem.

III. Line integrals involving conservative vector fields.

When this becomes applicable we will have discussed conservative vector fields, path independence, and the curl of a vector field quite thoroughly. In any case, it may help to read C3M13 again. In the theorem stated there it is useful to note that part (a) is based on integrating along an elbow path $(x_0, y_0) \to (x, y_0) \to (x, y)$. Part (b) uses $(x_0, y_0, z_0) \to (x, y_0, z_0) \to (x, y, z_0) \to (x, y, z)$.

Example 2 of C3M13, revisited Evaluate $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ if

$$\overrightarrow{F}(x, y) = \left\langle 3x^2 \cos(2y) + \frac{y}{x},\ -2x^3 \sin(2y) + \ln x \right\rangle$$

and C is the polygonal path $(1, \pi/4)$ to $(4, \pi/6)$ to $(3, \pi/3)$.

We note that $P = (1, \pi/4)$ and $Q = (3, \pi/3)$ and recall that $g(x, y) = x^3 \cos(2y) + y \ln x$ is a potential function for $\overrightarrow{F}$.

$$\begin{aligned}\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= g(Q) - g(P) = g(3, \pi/3) - g(1, \pi/4) \\ &= \left(27\cos\left(\frac{2\pi}{3}\right) + \frac{\pi}{3}\ln 3\right) - \left(\cos\left(\frac{\pi}{2}\right) + \frac{\pi}{4}\ln 1\right) \\ &= \frac{-27}{2} + \frac{\pi}{3}\ln 3\end{aligned}$$

The theorem that follows sums up conservative vector fields and line integrals quite nicely.

THEOREM: Suppose $\overrightarrow{F}$ is a vector field with all relevant components and their partial derivatives continuous on an open simply connected and connected region D. Then the following are equivalent (TFAE):

(A) $\overrightarrow{F}$ is conservative.

(B) $\overrightarrow{F}$ is path independent.

(C) $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for all closed paths C.

(D) The curl of $\overrightarrow{F}$, $\nabla \times \overrightarrow{F} = \overrightarrow{0}$ at all points of D.

Further, if g is a potential function for $\overrightarrow{F}$ on D and C is a path with initial point P and terminal point Q, then

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = g(Q) - g(P).$$

EXERCISES

I. Evaluate the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ over the given path.

1. $\overrightarrow{F}(x, y) = \langle 3y/x,\ 5x \rangle$ and C is parameterized by $\overrightarrow{\alpha}(t) = \langle t^2, t^3 \rangle$ for $1 \le t \le 2$.

2. C is the straight line segment from $P(-1, 3, 1)$ to $Q(1, 2, 3)$, find

$$\int_C y\,dx - z\,dy + 2y\,dz$$

3. C is the path that traverses the circle $x^2+y^2=1$ from $P(0,-1)$ to $Q(-1,0)$ in the counterclockwise direction. Evaluate $\int_C x^2\,dx + y\,dy$.

4. C is the straight line path from $P(2,-1,3)$ to $Q(-1,1,1)$. $\overrightarrow{F}(x,y,z) = \langle x-y-z,\ 2x+y-2z,\ x+y\rangle$.

II. Determine whether or not the given integral is path independent.

$$\int_P^Q (2y^3)\,dx + (6xy^2+z^2)\,dy + (2yz)\,dz$$

III. For the functions given:

(A) Prove that $\overrightarrow{F}$ is path independent

(B) Find a potential function for $\overrightarrow{F}$

(C) Evaluate $\int_C \overrightarrow{F}\cdot d\overrightarrow{r}$ along the path C

1. $\overrightarrow{F}(x,y) = \langle 4x^3\ln y + y,\ \frac{x^4}{y} + x\rangle$ with C the polygonal path $(1,1)$ to $(5,2)$ to $(3,e)$.

2. $\overrightarrow{F}(x,y) = \langle 6x^2\ln y,\ \frac{2x^3}{y} + 1\rangle$, C the polygonal path $(0,1)$ to $(1,2)$ to $(4,3)$

3. $\overrightarrow{F}(x,y) = \left\langle 4\cos(2x)\ln(y) + 6xy^3 + 1,\ \frac{2\sin(2x)}{y} + 9x^2y^2\right\rangle$ with C the line segment from $(\pi/6,1)$ to $(\pi/4,2e)$.

4. $\overrightarrow{F}(x,y,z) = \langle 2xy+\cos(\pi z),\ x^2+z,\ y-\pi x\sin(\pi z)\rangle$ with C the line segment from $P(1,1,1)$ to $Q(2,2,2)$.

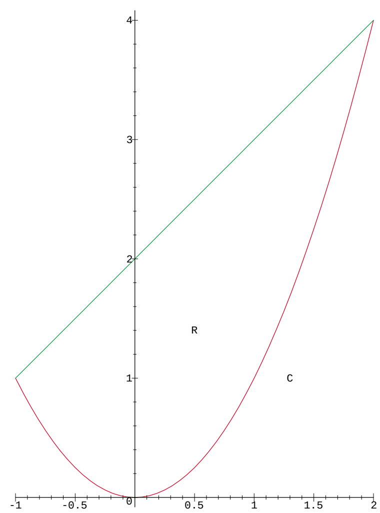
C3M15

Notes on Line Integrals - Green's Theorem

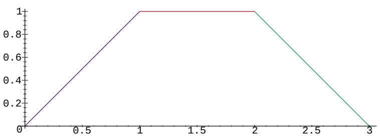
GREEN'S THEOREM: Let R be a simple region in the xy-plane with a piecewise smooth boundary C that is oriented counterclockwise. Let $\overrightarrow{F}$ be a vector field with all relevant components and their partial derivatives continuous on an open region containing R. Then

$$\oint_C \overrightarrow{F}\cdot d\overrightarrow{r} = \oint_C M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Example of a typical simple region in the plane with a piecewise smooth boundary



Example 1 Evaluate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ for $\overrightarrow{F}(x,y) = \langle 12x^2 \sin y + 3xy^2,\, 4x^3 \cos y + 6x^2 y\rangle$ and the polygonal path $(0,0) \to (3,0) \to (2,1) \to (1,1) \to (0,0)$.



We form the integrand for the double integral in Green's Theorem.

$$\frac{\partial N}{\partial x} = 12x^2 \cos y + 12xy \quad \text{and} \quad \frac{\partial M}{\partial y} = 12x^2 \cos y + 6xy \quad \Longrightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 6xy$$

It follows from Green's Theorem that

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\, dA \\
&= \int_0^1 \int_y^{3-y} 6xy \, dx \, dy \\
&= \int_0^1 3x^2 y \Big|_y^{3-y} dy = \int_0^1 3(3-y)^2 y - 3y^3 \, dy \\
&= \int_0^1 27y - 18y^2 \, dy = \frac{27}{2} y^2 - 6y^3 \Big|_0^1 \\
&= \frac{15}{2}
\end{aligned}$$

Example 2 (Example 2 of C3M14 revisited) The vector-valued function was $\overrightarrow{F}(x,y) = \langle -y, x\rangle$. Please refer back for a diagram of the region. With $M(x,y) = -y$ and $N(x,y) = x$,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2$$

So we have

$$\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_R 2\, dA = (2)\text{area}(R) = (2)\left(\frac{\pi}{2} + 2\right) = \pi + 4$$

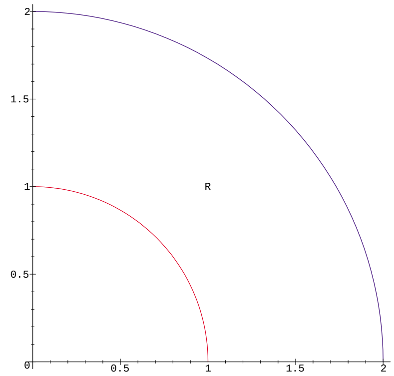
The computation of the area was easy - one half of a disk of radius 1 and one half of a square of side 2.

Example 3 The region R is that portion of the first quadrant between the circles of radius 1 and 2 centered at the origin. The vector-valued function is

$$\overrightarrow{F}(x,y) = \langle 4 + e^{\sqrt{x}}, \sin(y) + 3x^2\rangle$$

and the objective is to evaluate $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r}$ if C is the boundary of R traversed in a counterclockwise manner. It is not easy to evaluate the line integral around this path, so our approach is to use Green's theorem. We also observe that polar coordinates provides the simplest double integral. First, we will display the region.

```
> with(student):    with(plots):
> Q1:=polarplot([1,t,t=0..Pi/2]):    Q2:=polarplot([2,t,t=0..Pi/2]):
> T:=textplot([1,1,'R']):
> display(Q1,Q2,T);
```




```
> F:=(x,y)->[4+exp(sqrt(x)),sin(y)+3*x^2];
```

$$F := (x, y) \to [4 + e^{\sqrt{x}}, \sin(y) + 3x^2]$$

```
> Nx:=diff(F(x,y)[2],x);
```

$$Nx := 6x$$

```
> My:=diff(F(x,y)[1],y);
```

$$My := 0$$

```
> grand:=Nx-My;
```

$$grand := 6x$$

```
> grand:=subs(x=r*cos(t),grand);
```

$$grand := 6\, r\, \cos(t)$$

```
> Ans:=Doubleint(grand*r,r=1..2,t=0..Pi/2);
```

$$Ans := \int_0^{\pi/2} \int_1^2 6r^2 \cos(t)\, dr\, dt$$

```
> Greenans:=value(Ans);
```

$$Greenans := 14$$

Example 4 If $\overrightarrow{F}(x, y) = \langle -y/2, x/2 \rangle$, then $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = \dfrac{1}{2} - \left(-\dfrac{1}{2}\right) = 1$ and

$$\iint_R 1\, dA = \text{area}(R)$$

gives us a way of using line integrals to determine the area of a region. If the objective is to find the area of a region whose boundary is reasonable to parameterize, then **choose** the vector-valued function $\overrightarrow{F}(x, y) = \langle -y/2, x/2 \rangle$ and apply Green's Theorem by evaluating the resulting line integral.

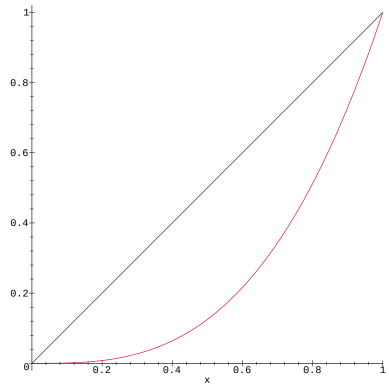
Suppose we find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. We parameterize C in the counterclockwise direction by $\overrightarrow{\alpha}(t) = \langle a \cos t, b \sin t \rangle$ for $0 \le t \le 2\pi$. With $\overrightarrow{F}(x, y) = \langle -y/2, x/2 \rangle$ we have

$$\begin{aligned}
\overrightarrow{F}\big(\overrightarrow{\alpha}(t)\big) &= \langle -\frac{b}{2} \sin t,\ \frac{a}{2} \cos t \rangle \\
\overrightarrow{\alpha}'(t) &= \langle -a \sin t,\ b \cos t \rangle \\
\overrightarrow{F}\big(\overrightarrow{\alpha}(t)\big) \cdot \overrightarrow{\alpha}'(t) &= \frac{ab}{2} \sin^2 t + \frac{ab}{2} \cos^2 t = \frac{ab}{2} \\
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} \frac{ab}{2}\, dt \\
&= \pi ab
\end{aligned}$$

Thus, the area of the ellipse is πab. What happens if $a = b$?

Example 5 Verify Green's Theorem for $\overrightarrow{F}(x, y) = \langle y^3, x^3 + 3xy^2 \rangle$ and the region R which lies between $y = x$ and $y = x^3$.

```
> with(student):    with(plots):    with(linalg):
> plot([x^3,x],x=0..1);
```




```
> M:=(x,y)->y^3;  N:=(x,y)->x^3+3*x*y^2;
```

$$M := (x, y) \to y^3$$
$$N := (x, y) \to x^3 + 3xy^2$$

```
> F:=(x,y)->[M(x,y),N(x,y)];
```

$$F := (x, y) \to [M(x, y), N(x, y)]$$

Let's begin by evaluating the line integral along the cubic path.

```
> alpha:=[t,t^3];
```

$$\alpha := [t, t^3]$$

```
> Falpha:=F(op(alpha));
```

$$Falpha := [t^9, t^3 + 3t^7]$$

```
> alphaprime:=diff(alpha,t);
```

$$alphaprime := [1, 3t^2]$$

```
> grand1:=innerprod(Falpha,alphaprime);
```

$$grand1 := t^9 + 3(t^3 + 3t^7)t^2$$

```
> Lint1:=Int(grand1,t=0..1);
```

$$Lint1 := \int_0^1 t^9 + 3(t^3 + 3t^7)t^2 \, dt$$

```
> V1:=value(Lint1);
```

$$V1 := \frac{3}{2}$$

For our second path, we must start at $(1, 1)$ and end at $(0, 0)$ along $y = x$. Note how we reverse the usual parameterization.

```
> beta:=[1-t,1-t];
```

$$\beta := [1 - t, 1 - t]$$

```
> Fbeta:=F(op(beta));
```

$$Fbeta := [(1 - t)^3, 4(1 - t)^3]$$

```
> betaprime:=diff(beta,t);
```

$$betaprime := [-1, -1]$$

```
> grand2:=innerprod(Fbeta,betaprime);
```

$$grand2 := 5(-1 + t)^3$$

```
> Lint2:=Int(grand2,t=0..1);
```

$$Lint2 := \int_0^1 5(-1 + t)^3 \, dt$$

```
> V2:=value(Lint2);
```

$$V2 := \frac{-5}{4}$$

```
> Lintanswer:=V1+V2;
```

$$Lintanswer := \frac{1}{4}$$

From this we may conclude that $\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = \frac{1}{4}$. Now we will evaluate $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$ and compare the answers.

```
> Nx:=diff(N(x,y),x);  My:=diff(M(x,y),y);
```

$$Nx := 3x^2 + 3y^2$$

$$My := 3y^2$$

```
> grandGT:=Nx-My;
```

$$grandGT := 3x^2$$

```
> ansGT:=Doubleint(grandGT,y=x^3..x,x=0..1);
```

$$ansGT := \int_0^1 \int_{x^3}^{x} 3x^2\,dy\,dx$$

```
> GTanswer:=value(ansGT);
```

$$GTanswer := \frac{1}{4}$$

And as we expected, the two answers agree.

EXERCISES

I. Evaluate the line integrals directly.

1. $\int_C (x^2 - y^3)\,dx + (x^2 + y^2)\,dy$, $C : (1,0) \to (0,1)$ on $x^2 + y^2 = 1$.
2. $\int_C (x^2 - y^2)\,dy$, $C : (0,0) \to (1,2)$ on $y = 2x^2$.
3. $\int_C (x + 2y)\,dx + y\,dy$, C counterclockwise on the ellipse $x^2 + 4y^2 = 1$
4. $\int_{(1,0)}^{(0,1)} y(e^{xy} + 1)\,dx + x(e^{xy} + 1)\,dy$

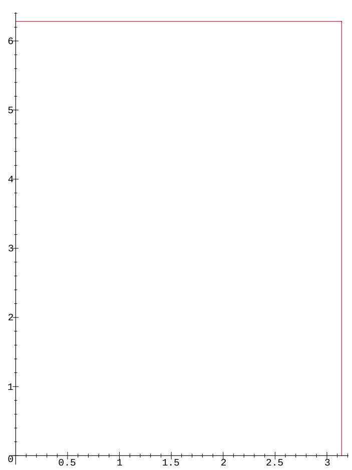
II. Use Green's Theorem to evaluate the line integrals.

5. $\int_C (x^2 y + y^3 - \ln x)\,dx + (x\cos y + xy^2)\,dy$, where C is the polygonal path $(1,1) \to (2,1) \to (2,2) \to (1,2) \to (1,1)$
6. $\int_C (\sin y - x^2 y)\,dx + (x\cos y + xy^2)\,dy$, C is the circle $x^2 + y^2 = 1$ counterclockwise.
7. $\int_C \left(e^x \sin y - xy^2\right)dx + \left(e^x \cos y + x^2 y\right)dy$, C is the ellipse $9x^2 + 4y^2 = 36$ counterclockwise.

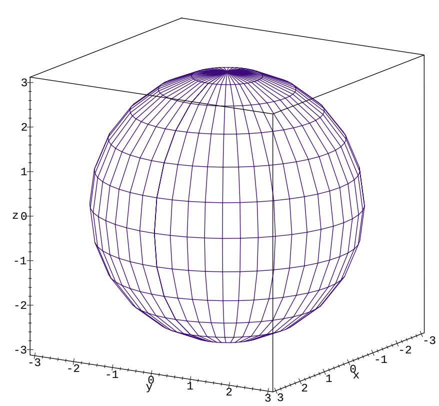
Surface Area Revisited and Surface Integrals

Now that we have some experience with triple integrals in cylindrical and spherical coordinates, we may expand the surfaces that we can parameterize to include spheres. Cylinders were discussed in Example 2 of C3M10.

Example 1 Let's parameterize a sphere S of radius ρ and then compute $d\sigma$. We will use a rectangle $R = \{(\varphi, \theta) : 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$ as the domain of g.



$\xrightarrow{g}$



$$g(\theta,\varphi) = \begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\varphi \end{pmatrix} \qquad \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}$$

And

$$\frac{\partial g}{\partial \theta} = \begin{pmatrix} -\rho \sin\varphi \sin\theta \\ \rho \sin\varphi \cos\theta \\ 0 \end{pmatrix} \qquad \frac{\partial g}{\partial \varphi} = \begin{pmatrix} \rho \cos\varphi \cos\theta \\ \rho \cos\varphi \sin\theta \\ -\rho \sin\varphi \end{pmatrix}$$

The cross product (note the order) is

$$\begin{aligned} \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} &= \begin{vmatrix} \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \vec{i} & \vec{j} & \vec{k} \end{vmatrix} \\ &= \left\langle \rho^2\sin^2\varphi\cos\theta,\ \rho^2\sin^2\varphi\sin\theta,\ \rho^2\sin\varphi\cos\varphi\cos^2\varphi + \rho^2\sin\varphi\cos\varphi\sin^2\theta \right\rangle \\ &= \left\langle \rho^2\sin^2\varphi\cos\theta,\ \rho^2\sin^2\varphi\sin\theta,\ \rho^2\sin\varphi\cos\varphi \right\rangle \end{aligned}$$

So we have

$$\left\| \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} \right\| = \rho^2 \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^{2+\sin}}$$

$\sqrt{}$

$q \quad \rho$

$\varphi \cos \quad {}^2 \quad \sin \quad \varphi \quad \varphi$

$4 \quad 2$

EXAMPLE 2 Let's compute the surface integral $\iint_S e^z\, d\sigma$ where S is the sphere of radius a. We may let $d\sigma = a^2 \sin\varphi\, d\varphi\, d\theta$ from what we found in the preceding example. Thus

$$\begin{aligned}\iint_S e^z\, d\sigma &= \int_0^{2\pi}\int_0^{\pi} e^{a\cos\varphi} a^2 \sin\varphi\, d\varphi\, d\theta \qquad \begin{pmatrix} u = a\cos\varphi \\ du = -a\sin\varphi\, d\varphi \end{pmatrix} \\ &= a\int_0^{2\pi} -e^{a\cos\varphi}\Big|_{\varphi=0}^{\varphi=\pi} d\theta \\ &= a\int_0^{2\pi} -e^{-a} + e^a\, d\theta \\ &= 2\pi a\big(e^a - e^{-a}\big)\end{aligned}$$

THE UNIT NORMAL VECTOR $\overrightarrow{n}(u,v)$. Earlier we found $\dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v}$, the fundamental cross product, and mentioned that it is a vector that is orthogonal or normal to the surface S. But $\dfrac{\partial g}{\partial v} \times \dfrac{\partial g}{\partial u} = -\dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v}$ is also normal to the surface and is opposite in direction. At this point we must read about oriented surfaces or listen to what the instructor says about them. Remember, the Möbius strip is not orientable while the surface of a sphere is orientable. Consequently, we may refer to the "outward" normal, which means that a minus sign may be involved implicitly. In any case, we write

$$\overrightarrow{n} = \frac{\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}}{\left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\|}$$

which is a unit normal to the surface.

As in RECALL, we assume that a vector field $\overrightarrow{F}$ is defined on the surface S parameterized by g on D. In C3M10 we defined $d\sigma = \left\| \dfrac{\partial g}{\partial u} \times \dfrac{\partial g}{\partial v} \right\| du\, dv$. A parallelogram was alluded to then, but was not specified. Consider the parallelogram generated by the two vectors, $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$, at the point $g(u,v)$ on the surface S. Each is tangent to S and so is this parallelogram that they generate. By multiplying the first vector by du and the second by dv, a smaller parallelogram is determined and the length of their cross product is the area of this parallelogram. This length is regarded as an element of area, just like $dx\, dy$ was when double integrals were introduced.

$$\begin{aligned} dS &= \frac{\partial g}{\partial u}\, du \times \frac{\partial g}{\partial v}\, dv \\ &= \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\, du\, dv \\ &= \overrightarrow{n} \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\, dv \\ &= \overrightarrow{n}\, d\sigma \\ \text{and} \qquad d\sigma &= \|dS\| \end{aligned}$$

Suppose that $\overrightarrow{F}$ is a vector-valued function and that a path Γ and a surface Σ are in its domain. If Γ is parameterized by $\overrightarrow{\alpha}(t)$ and $\overrightarrow{T} = \overrightarrow{\alpha}'(t)/\|\overrightarrow{\alpha}'(t)\|$ is the unit tangent, then $\overrightarrow{T}\, ds$ and ds for line integrals are analogous to dS and $d\sigma$, respectively, for surface integrals. While $\overrightarrow{F} \cdot \overrightarrow{T}$ provided the component of $\overrightarrow{F}$ tangential to the path, $\overrightarrow{F} \cdot \overrightarrow{n}$ will provide the normal component of $\overrightarrow{F}$ at the surface. We may interpret this last entity as rate of flow outward at the surface if the function $\overrightarrow{F}$ represents fluid flow. For this reason we define the

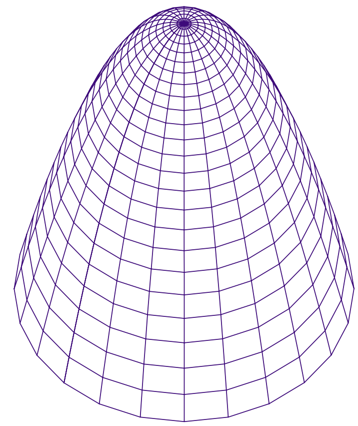
Flux Integral of $\overrightarrow{F}$ over S

$$\begin{aligned}\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, d\sigma &= \iint_D \overrightarrow{F}\big(g(u,v)\big) \cdot \overrightarrow{n} \left\| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right\| du\, dv \\ &= \iint_D \overrightarrow{F}\big(g(u,v)\big) \cdot \left(\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right) du\, dv\end{aligned}$$

which simplifies because of $\left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\|$ in both numerator and denominator. Actually, we will be more concerned about integrals of this type when the vector-valued function is the CURL of $\overrightarrow{F}$, i.e. $\nabla \times \overrightarrow{F}$.

EXAMPLE 3 (to be revisited)

Integrate the normal component of the curl of $\overrightarrow{F}(x,y,z) = \langle y^2, x, -xz\rangle$ over the surface $S = \{(x,y,z) : z = 9 - x^2 - y^2, z \geq 0\}$. That is, evaluate



$$\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$$

First,

$$\nabla \times \overrightarrow{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & -xz \\ \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \end{vmatrix} = \langle 0, z, 1-2y\rangle$$

We parametrize S as a function of r and θ over the rectangle $D = \{(r,\theta) : 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

$$g(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 9-r^2 \end{pmatrix} \qquad \frac{\partial g}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ -2r \end{pmatrix} \qquad \frac{\partial g}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

And we compute the other components of the integral, beginning with the curl of $\overrightarrow{F}$ evaluated at $g(r,\theta)$.

$$\begin{aligned}
(\nabla \times \overrightarrow{F})\big(g(r,\theta)\big) &= \langle 0, 9-r^2, 1-2r\sin\theta\rangle \\
\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} &= \begin{vmatrix} \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \\ \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \end{vmatrix} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r\rangle \\
(\nabla \times \overrightarrow{F})\big(g(r,\theta)\big) \cdot \left(\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta}\right) &= (9-r^2)(2r^2\sin\theta) + r - 2r^2\sin\theta \\
&= 16r^2\sin\theta - 2r^4\sin\theta + r
\end{aligned}$$

Putting this all together we have

$$\begin{aligned}
\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma &= \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} (16r^2 - 2r^4)\sin\theta + r\, d\theta\, dr \\
&= \int_{r=0}^{r=3} -(16r^2 - 2r^4)\cos\theta + r\theta \Big|_{\theta=0}^{\theta=2\pi} dr \\
&= \int_0^3 2\pi r\, dr = \pi r^2 \Big|_0^3 \\
&= 9\pi
\end{aligned}$$

We will return to this example very soon.

C3M16

Stoke's Theorem

STOKE'S THEOREM: The line integral of the tangential component of $\overrightarrow{F}$ taken along the boundary C of the surface S once in the positive direction equals the surface integral of the normal component of the curl of $\overrightarrow{F}$ over S.

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$$

EXAMPLE 1 (Example 3 of Surface Integrals revisited) We return to our example of the paraboloid where $\overrightarrow{F}(x, y, z) = \langle y^2, x, -xz \rangle$ and compute the line integral around the boundary $C = \{(x, y, z) : x^2 + y^2 = 9,\ z = 0\}$. Parametrize C by $\overrightarrow{\alpha}(t) = \langle 3\cos t, 3\sin t, 0 \rangle$ on the interval $[0, 2\pi]$.

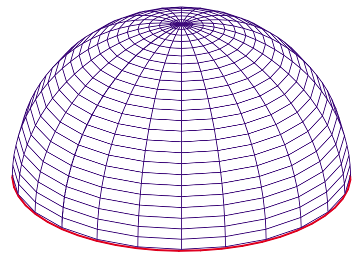
$$\begin{aligned}
\overrightarrow{F}\big(\overrightarrow{\alpha}(t)\big) &= \langle 9\sin^2 t, 3\cos t, 0 \rangle \\
\overrightarrow{\alpha}\,'(t) &= \langle -3\sin t, 3\cos t, 0 \rangle \\
\overrightarrow{F}\big(\overrightarrow{\alpha}(t)\big) \cdot \overrightarrow{\alpha}\,'(t) &= -27\sin^3 t + 9\cos^2 t \\
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} -27\sin^3 t + 9\cos^2 t \, dt \\
&= \int_0^{2\pi} -27(1 - \cos^2 t)\sin t + \frac{9}{2}(1 + \cos 2t)\, dt \\
&= 27\cos t - \frac{27}{3}\cos^3 t + \frac{9}{2}t + \frac{9}{4}\sin 2t \Big|_0^{2\pi} \\
&= 9\pi
\end{aligned}$$

AS IT SHOULD according to Stoke's Theorem.

provides the accumulative effect of the curl of $\overrightarrow{F}$ over the entire region D or S.

EXAMPLE 2 Suppose we are given the curl of $\overrightarrow{F}$, $\nabla \times \overrightarrow{F} = \langle 2y, -2z, 3\rangle$, but not the function $\overrightarrow{F}$ itself. Our surface is the upper hemisphere of radius 3. That is, $S = \{(x,y,z) : x^2+y^2+z^2 = 9, z \geq 0\}$. Our objective is to evaluate the surface integral $\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$.

(a) Use spherical coordinates to parameterize S



$$g(\theta,\varphi) = \begin{pmatrix} 3\sin\varphi\cos\theta \\ 3\sin\varphi\sin\theta \\ 3\cos\varphi \end{pmatrix} \qquad \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/2 \end{matrix}$$

We have

$$\overrightarrow{n} = -\left(\frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \varphi}\right) = \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} = \langle 9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi\rangle$$

$$(\nabla \times \overrightarrow{F})\big(g(\theta,\varphi)\big) = \langle 6\sin\varphi\sin\theta, -6\cos\varphi, 3\rangle$$

$$(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n} = 54\sin^3\varphi\sin\theta\cos\theta - 54\sin^2\varphi\cos\varphi\sin\theta + 27\sin\varphi\cos\varphi$$

Evaluating $\displaystyle\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$ we have

$$\int_0^{\pi/2}\int_0^{2\pi} 54\sin^3\varphi\sin\theta\cos\theta - 54\sin^2\varphi\cos\varphi\sin\theta + 27\sin\varphi\cos\varphi\, d\theta\, d\varphi = 27\pi$$

(b) Now we observe that the boundary of S is the circle
$C = \{(x,y,0) : x^2+y^2 = 9\}$, which just happens to be the boundary of the disk
$D = \{(x,y,z) : x^2+y^2 \leq 9, z = 0\}$. So we apply Stoke's theorem twice.

$$\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma = \oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_D (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$$

But on D, $\overrightarrow{n} = \overrightarrow{k} = \langle 0,0,1\rangle$ which means that $(\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n} = \langle 2y, 0, 3\rangle \cdot \langle 0,0,1\rangle = 3$ And we have

$$\iint_D 3\, dlh\alpha h \oint L\partial KAAwhLh\mathbf{L}\star Oh\star Ko\partial oOZoL\partial A\star\oint\alpha$$

$$G2 := [-r\sin(t), r\cos(t), 0]$$

```
> n:=crossprod(G1,G2);
```

$$n := [2r^2\cos(t), 2r^2\sin(t), \cos(t)^2 r + \sin(t)^2 r]$$

```
> grand:=innerprod(curlFatg,n);
```

$$grand := 2(25 - r^2)r^2\sin(t) + (1 - 2r\sin(t))(\cos(t)^2 r + \sin(t)^2 r)$$

```
> grand:=simplify(grand);
```

$$48r^2\sin(t) - 2r^4\sin(t) + r$$

```
> A:=Doubleint(grand,t=0..2*Pi,r=0..5);
```

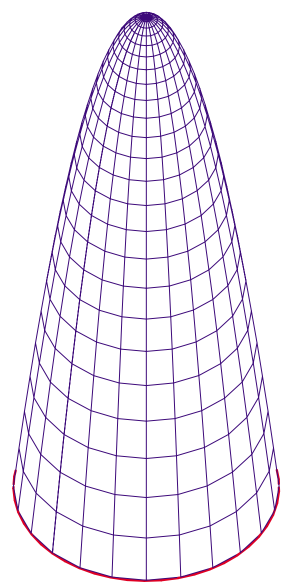
$$A := \int_0^5 \int_0^{2\pi} 48r^2\sin(t) - 2r^4\sin(t) + r\,dt\,dr$$

```
> Ans1:=value(A);
```

$$Ans1 := 25\pi$$

This is usually the more difficult answer to obtain when verifying Stoke's Theorem. Now, let's look at our surface.

```
> P1:=plot3d(g,r=0..5,t=0..2*Pi,color=blue):
> P2:=spacecurve([5*cos(t),5*sin(t),0],t=0..2*Pi,color=red):
> display(P1,P2);
```



Turning our attention to the circulation line integral, we begin by parameterizing the boundary which is shown in red in the actual plot.

```
> alpha:=[5*cos(t),5*sin(t),0];
```

$$\alpha := [5\cos(t), 5\sin(t), 0]$$

```
> Falpha:=subs(x=alpha[1],y=alpha[2],z=alpha[3],F);
```

$$Falpha := [25\sin(t)^2, 5\cos(t), 0]$$

```
> alphaprime:=diff(alpha,t);
```

$$alphaprime := [-5\sin(t), 5\cos(t), 0]$$

```
> grand1:=innerprod(Falpha,alphaprime);
```

$$grand1 := -125\sin(t)^3 + 25\cos(t)^2$$

```
> Lint:=Int(grand1,t=0..2*Pi);
```

$$Lint := \int_0^{2\pi} -125\sin(t)^3 + 25\cos(t)^2\,dt$$

```
> Ans2:=value(Lint);
```

$$Ans2 := 25\pi$$

And we see that our answers agree, as they should.

C3M16 Problems

Verify Stoke's Theorem for:

1. $\overrightarrow{F}(x,y,z) = \langle 2z, 3x, 4y\rangle$, $\quad S = \{(x,y,z) : 0 \le z = 9 - x^2 - y^2\}$.
2. $\overrightarrow{F}(x,y,z) = \langle 4x^2 + y, 2y, x - 3z^2\rangle$, $\quad S = \{(x,y,z) : 3x + y + 3z = 4, x, y, z \ge 0\}$.

Apply Stoke's Theorem to evaluate $\displaystyle\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$:

3. $\overrightarrow{F}(x,y,z) = \langle 0, x - y, z^3\rangle$ $\quad S$ is the surface of the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$
4. Let the surface S be the curved side of the cylinder $x^2 + y^2 = 9$, $0 \le z \le 2$ together with the top, $x^2 + y^2 \le 9, z = 2$. $\overrightarrow{F}(x,y,z) = \langle -y, x, x^2\rangle$.

C3M16 Maple Problems Use Maple in the remaining problems.

5. Apply Stoke's Theorem so as to evaluate $\displaystyle\iint_S (\nabla \times \overrightarrow{F}) \cdot \overrightarrow{n}\, d\sigma$ if $\overrightarrow{F}(x,y,z) = \langle 0, x - y, z^3\rangle$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$
6. Apply Stoke's Theorem so as to evaluate the line integral $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$. C is the square with vertices $P_1(0,0,4)$, $P_2(3,0,4)$, $P_3(3,2,4)$, $P_4(0,2,4)$ taken in that order and $\overrightarrow{F}(x,y,z) = \langle y/z, x^2 y, x + z\rangle$.

C3M17

The Divergence Theorem

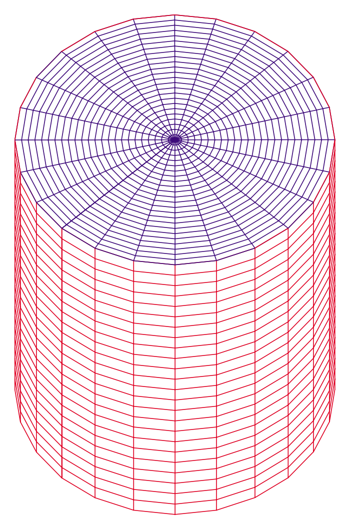
THE DIVERGENCE THEOREM: Let Q be a simple solid region whose boundary surface S is oriented by the unit normal $\overrightarrow{n}$ directed outward from Q, and let $\overrightarrow{F}$ be a vector field whose component functions have continuous partial derivatives on Q. Then

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, d\sigma = \iiint_Q \nabla \cdot \overrightarrow{F}\, dV$$

This states that the flux through a closed surface equals the total divergence throughout the solid.

EXAMPLE 1: Consider the solid cylinder
$Q = \{(x,y,z) : x^2 + y^2 \le 1, 0 \le z \le 1\}$ and
the vector field $\overrightarrow{F}(x,y,z) = \langle xz, y, z\rangle$.
The divergence of $\overrightarrow{F}$ is $\nabla \cdot \overrightarrow{F} = z + 1 + 1 = z + 2$.

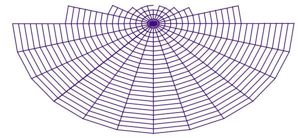
$$\begin{aligned}
\iiint_Q \nabla \cdot \overrightarrow{F}\, dV &= \int_0^{2\pi}\int_0^1\int_0^1 (z+2) r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(\frac{z^2}{2} + 2z\right) r \Big|_{z=0}^{z=1} dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{5}{2} r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{5}{4} r^2 \Big|_0^1 d\theta \\
&= \int_0^{2\pi} \frac{5}{4}\, d\theta \\
\iiint_Q \nabla \cdot \overrightarrow{F}\, dV &= \frac{5\pi}{2}
\end{aligned}$$



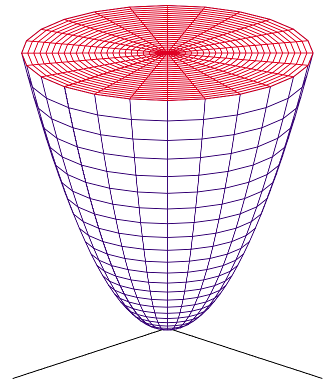
Now we will compare our answer with the sum of three surface integrals, each with the normal vector $\overrightarrow{n}$ pointing outward.

$\underline{\text{bottom}}$ $\quad z = 0, \quad \overrightarrow{n} = -\overrightarrow{k}, \quad \overrightarrow{F}(x, y, 0) = \langle 0, y, 0 \rangle, \quad \overrightarrow{F} \cdot \overrightarrow{n} = 0$ so

$$\iint_{\text{bottom}} \overrightarrow{F} \cdot \overrightarrow{n} \, d\sigma = 0$$



EXAMPLE 2 Let's compare the values of the integrals of the Divergence theorem for the vector field $\overrightarrow{F}(x, y, z) = \langle 1, 1, z \rangle$ for the solid Q that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.



Volume integral

$\nabla \cdot \overrightarrow{F} = 1$ so

$$\iiint_Q \nabla \cdot \overrightarrow{F}\, dV = \iiint_Q 1\, dV$$
$$= \int_{-2}^{2} \int_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} \int_{z=x^2+y^2}^{z=4} 1\, dz\, dx\, dy$$
$$= \int_0^{2\pi} \int_{r=0}^{r=2} \int_{z=r^2}^{z=4} r\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_{r=0}^{r=2} 4r - r^3\, dr\, d\theta$$
$$= \int_0^{2\pi} 4\, d\theta$$
$$\iiint_Q \nabla \cdot \overrightarrow{F}\, dV = 8\pi$$

Surface integrals

(a) Top $\quad z = 4 \quad, \overrightarrow{n} = \overrightarrow{k} = \langle 0, 0, 1 \rangle, \quad \overrightarrow{F}(x, y, 4) = \langle 1, 1, 4 \rangle, \quad \overrightarrow{F} \cdot \overrightarrow{n} = 4$

$$\iint_S 4\, d\sigma = 4(4\pi) = 16\pi$$

(b) Bottom $\quad z = x^2 + y^2$, but we will parametrize in polar coordinates.

$$g(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{pmatrix} \qquad \begin{matrix} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{matrix}$$
$$\frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$$

It is important to note that our normal vector points inward, so we must choose its negative! Thus

$$\overrightarrow{n} = \langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle$$

We have $\overrightarrow{F}\big(g(r, \theta)\big) = \langle 1, 1, r^2 \rangle, \qquad \overrightarrow{F} \cdot \overrightarrow{n} = 2r^2\cos\theta + 2r^2\sin\theta - r^3$, so

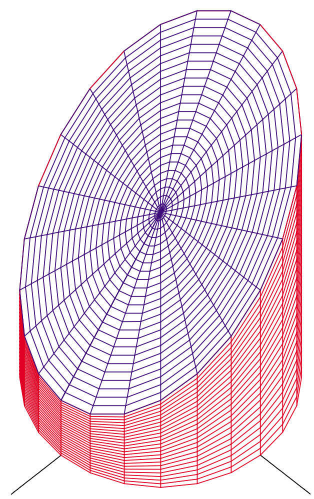
$$\int_0^{2\pi} \int_0^2 2r^2\cos\theta + 2r^2\sin\theta - r^3\, dr\, d\theta = \int_0^{2\pi} \frac{2r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta - \frac{r^4}{4}\Big|_0^2\, d\theta$$
$$= \int_0^{2\pi} \frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta - 4\, d\theta$$
$$= -8\pi$$

And as was predicted,

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, d\sigma = 16\pi - 8\pi = 8\pi.$$

Example 3, Maple: We will illustrate the equivalence of the two integrals in the Divergence Theorem with a challenging problem. The solid is bounded by the oblique plane $2x + z = 6$, the vertical cylinder $x^2 + y^2 = 4$, and the xy-plane. The vector function is $\overrightarrow{F}(x, y, z) = \langle x^2 + z, xy + 2z, 3y \rangle$. We will begin by sketching the solid, then compute the divergence over the solid, and end with the three surface integrals. The easiest approach is to use cylindrical coordinates.

```
> restart:   with(student):   with(plots):   with(linalg):
> S:=cylinderplot([2,theta,z],theta=0..2*Pi,z=0..6-2*2*cos(theta),color=red):
> T:=cylinderplot([r,theta,6-2*r*cos(theta)],r=0..2,theta=0..2*Pi,color=blue):
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> display(S,T,xaxis,yaxis);
```



```
> F:=(x,y,z)->[x^2+z,x*y+2*z,3*y];
```

$$F := (x, y, z) \to [x^2 + z, xy + 2z, 3y]$$

```
> divF:=diverge(F(x,y,z),[x,y,z]);
```

$$divF := 3x$$

```
> grand:=subs(x=r*cos(theta),y=r*sin(theta),z=z,divF);
```

$$grand := 3r\cos(\theta)$$

```
> divint:=Tripleint(grand*r,z=0..6-2*r*cos(theta),r=0..2,theta=0..2*Pi);
```

$$divint := \int_0^{2\pi} \int_0^2 \int_0^{6-2r\cos(\theta)} 3r^2 \cos(\theta)\, dz\, dr\, d\theta$$

```
> divanswer:=value(divint);
```

$$divanswer := -24\pi$$

It should be obvious that we have just finished the easy half of this problem. We will work from top to bottom on the surfaces. We begin with the oblique plane. Note how u plays the role of r and v that of θ.

```
> g1:=[u*cos(v),u*sin(v),6-2*u*cos(v)];
```

$$g1 := [u\cos(v), u\sin(v), 6 - 2u\cos(v)]$$

```
> g1u:=map(diff,g1,u);
```

$$g1u := [\cos(v), \sin(v), -2\cos(v)]$$

```
> g1v:=map(diff,g1,v);
```

$$g1v := [-u\sin(v), u\cos(v), 2u\sin(v)];$$

```
> fcp1:=crossprod(g1u,g1v);
```

$$fcp1 := [2\sin(v)^2 u + 2\cos(v)^2 u, 0, \cos(v)^2 u + \sin(v)^2 u]$$

```
> fcp1:=simplify(op(fcp1),symbolic);
```

$$fcp1 := [2u, 0, u]$$

```
> Fatg1uv:=F(op(g1));
```

$$Fatg1uv := [u^2\cos(v)^2 + 6 - 2u\cos(v), u^2\cos(v)\sin(v) + 12 - 4u\cos(v), 3u\sin(v)]$$

```
> grand1:=innerprod(Fatg1uv,fcp1);
```

$$grand1 := 2u^3\cos(v)^2 + 12u - 4u^2\cos(v) + 3u^2\sin(v)$$

```
> ans1:=Doubleint(grand1,u=0..2,v=0..2*Pi);
```

$$ans1 := \int_0^{2\pi}\int_0^2 2u^3\cos(v)^2 + 12u - 4u^2\cos(v) + 3u^2\sin(v)\,du\,dv$$

```
> flux1:=value(ans1);
```

$$flux1 := 56\pi$$

Now we will work on the curved side.

```
> g2:=vector([2*cos(u),2*sin(u),v]);
```

$$g2 := [2\cos(u), 2\sin(u), v]$$

```
> g2u:=map(diff,g2,u);
```

$$g2u := [-2\sin(u), 2\cos(u), 0]$$

```
> g2v:=map(diff,g2,v);
```

$$g2v := [0, 0, 1]$$

```
> fcp2:=crossprod(g2u,g2v);
```

$$fcp2 := [2\cos(u), 2\sin(u), 0]$$

```
> Fatg2uv:=F(op(g2));
```

$$Fatg2uv := [4\cos(u)^2 + v, 4\cos(u)\sin(u) + 2v, 6\sin(u)]$$

```
> grand2:=innerprod(Fatg2uv,fcp2);
```

$$grand2 := 8\cos(u)^3 + 2\cos(u)v + 8\cos(u)\sin(u)^2 + 4\sin(u)v$$

```
> grand2:=simplify(grand2,symbolic);
```

$$grand2 := 2\cos(u)v + 8\cos(u) + 4\sin(u)v$$

```
> ans2:=Doubleint(grand2,v=0..6-2*2*cos(u),u=0..2*Pi);
```

$$ans2 := \int_0^{2\pi}\int_0^{6-4\cos(u)} 2\cos(u)v + 8\cos(u) + 4\sin(u)v\,dv\,du$$

```
> flux2:=value(ans2);
```

$$flux2 := -80\pi$$

Let's think about the bottom surface for a moment. The *outward* unit normal vector is $-\overrightarrow{k} = \langle 0, 0, -1\rangle$.

```
> Fonbot:=F(x,y,0);
```

$$Fonbot := [x^2, xy, 3y]$$

```
> n:=vector([0,0,-1]);
```

$$n := [0, 0, -1]$$

```
> grand3:=innerprod(Fonbot,n);
```

$$grand3 := -3y$$

```
> grand3:=subs(x=r*cos(theta),y=r*sin(theta),grand3);
```

$$grand3 := -3r\sin(\theta)$$

```
> ans3:=Doubleint(grand3*r,r=0..2,theta=0..2*Pi);
```

$$ans3 := \int_0^{2\pi}\int_0^2 -3r^2\sin(\theta)\,dr\,d\theta$$

```
> flux3:=value(ans3);
```

$$flux3 := 0$$

```
> Fluxtotal:=flux1+flux2+flux3;
```

$$Fluxtotal := -24\pi$$

Once again, the divergence integral is seen to be the same as the flux integral. At this point, the reader should begin to develop an appreciation for how the divergence integral is usually easier to evaluate than the surface integral(s).

C3M17 problems

1. Use Maple to evaluate the flux integral, $\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,d\sigma = \iint_D \overrightarrow{F}(g(u,v))\cdot\left(\frac{\partial g}{\partial u}\times\frac{\partial g}{\partial v}\right)du\,dv$ directly. $\overrightarrow{F}(x,y,z) = \langle y, -x, 1\rangle$, $\overrightarrow{n}$ outward, S is the surface of the sphere $x^2+y^2+z^2=1$.

Use Maple and the Divergence Theorem to evaluate the given flux integrals, $\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,d\sigma$. (Do NOT evaluate the integrals directly!)

2. $\overrightarrow{F}(x,y,z) = \langle 3x^2, xy, z\rangle$, S bounds the solid $Q = \{(x,y,z) : x+y+z \leq 1,\ x,y,z \geq 0\}$

3. $\overrightarrow{F}(x,y,z) = \langle x^3, y^3, z^3\rangle$, Q is the solid above the cone $z^2 = x^2+y^2$ and inside the sphere $x^2+y^2+z^2 = 9$. Hint: use spherical coordinates.

Acknowledgement: Some of the problems for these notes have been borrowed from various texts. The authors include W. F. Trench, Earl Swokowski, and several others.

SUMMARY OF MAPLE COMMANDS

Using the package ‘student’

Maple Command	**Output**
`area:=Pi*r^2;`	$area := \pi r^2$ Sets πr^2 as an expression in r named $area$
`f:=x->sqrt(4+9*x^2);`	$f := x \to \sqrt{4+9x^2}$ Sets $\sqrt{4+9x^2}$ as a function named $f(x)$
`subs(r=5,area);`	25π Substitutes 5 into the expression $area$ as r
`f(1);`	$\sqrt{13}$ Evaluates the function f at 1
`plot(area,r=1..4);`	Produces a plot of expression $area$, r ranges from 1 to 4
`plot(f(x),x=-2..5);`	Produces a plot of function $f(x)$, x ranges from -2 to 5
`diff(area,r);`	$2\pi r$ The derivative of the expression $area$ with respect to r
`diff(f(x),x);`	$9\frac{x}{\sqrt{4+9x^2}}$ The derivative of the function $f(x)$ with respect to x
`int(area,r);`	$\frac{1}{3}\pi r^3$ The anti-derivative of the expression $area$ with respect to r
`int(area,r=1..4);`	21π The definite integral of $area$ from 1 to 4.
`eval(expr,x=a);`	Evaluates the expression *expr* at $x = a$
`evalf(%);`	65.97344573 The decimal form of the previous expression, 21π
`Int(area,r);`	$\int \pi r^2\, dr$ The inert expression whose value is the antiderivative of $area$
`sum(f(k),k=n1..n2);`	Evaluates the sum $\sum_{k=n1}^{k=n2} f(k)$ of $f(k)$ from $n1$ to $n2$
`leftbox(f(x),x=a..b,n);`	Displays the Riemann sum graphically for $f(x)$ over the interval $[a,b]$ using n intervals and left endpoints
`rightsum(f(x),x=a..b,n);`	Displays the Riemann sum in summation form for $f(x)$ over the interval $[a,b]$ using n intervals and right endpoints

Using the package ‘plots’

`spacecurve([f(t),g(t),h(t)],t=a..b);`	Displays the three-dimensional curve defined parametrically by three expressions in t, which ranges from a to b
`plot3d(expr,var1=a..b,var2=c..d`	Displays a surface for an expression in two variables
`plot3d([expr1,expr2,expr3],var1=a..b, var2=c,,d);`	Displays a surface parametrically as a function of two variables
`cylinderplot(expr,var1=a..b,var2=c..d);`	Displays $z =$expr in terms of $r =$var1 and $\theta =$var2
`polarplot(expr,var=a..b);`	Displays a polar coordinate plot of $r =$expr in terms of $\theta =$var
`sphereplot(expr,var1=a..b,var2=c..d);`	Displays $\rho =$expr in terms of $\theta =$var1 and $\phi =$var2
`cylinderplot([expr1,expr2,expr3], var1=a..b,var2=c..d);`	Displays a 3-d cylindrical plot parametrically, (r, θ, z)
`polarplot([expr1,expr2,var=a..b]);`	Displays a parametric polar coordinate plot of $r =$expr1 and $\theta =$expr2
`sphereplot([expr1,expr2,expr3], var1=a..b,var2=c..d);`	Displays a 3-d spherical plot parametrically, (ρ, θ, ϕ)
`implicitplot(equation,var1=a..b, var2=c..d);`	Displays a 2-d implicit plot of equation
`implicitplot3d(eqn,var1=a..b, var2=c..d,var3=e..f);`	Displays a 3-d plot for the equation eqn
`display({Plot1,Plot2,..Plotn});`	Displays n plots on the same coordinate system

Using the package ‘linalg’

`A:=vector([a,b,c]);`	Assigns the name A to the vector $\langle a, b, c\rangle$
`A[1]; A[2]; A[3];`	The components of the vector A are displayed
`F:=vector([expr1,expr2,expr3]);`	Assigns the name F to the vector expression
`map(diff,F,t);`	Gives the derivative of the vector $\overrightarrow{F}$ with respect to t
`innerprod(A,B);`	Finds $\overrightarrow{A} \cdot \overrightarrow{B}$, the scalar product of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$
`crossprod(A,B);`	Finds $\overrightarrow{A} \times \overrightarrow{B}$, the cross product of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$
`evalm(A+c*B);`	Finds $A + cB$ for vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, and scalar c
`hessian(f,[x,y]);`	Finds the hessian matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ for the function f of x and y
`det(A);`	Finds the determinant of the square matrix A
`subs({x=a,y=b},op(H));`	Substitutes values for x and y into components of matrix/vector H