Search and Detection

Bayes' Rule using Tables

Sometimes we find ourselves in the position of wondering "How did we get to where we are?" One way to get a probabilistic answer is by Bayes' Rule. Examples 2.38-2.39 in the textbook investigate a situation where a randomly chosen production item is found to be defective and we are interested in determining the (conditional) probability that it was produced by machine #3. Let's first set up the solution in a table form:

i	A | Bi	Bi	Bi & A	Bi | A
1	0.02000	0.30000	0.00600	0.24490
2	0.03000	0.45000	0.01350	0.55102
3	0.02000	0.25000	0.00500	**0.20408**
		1.00000	0.02450	1.00000

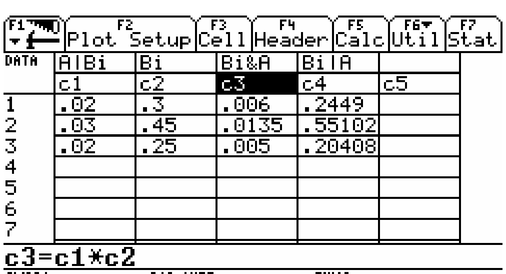
a) The column labeled **A | Bi** gives the conditional probabilities $P(A \mid B_i)$ where A is the event that the production item is defective and B_i is the event that machine $\# i$ produces the item, $i = 1,2,3$.

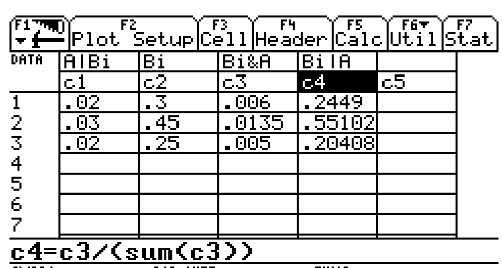
b) The next column **Bi** gives the probabilities $P(B_i)$ that each machine produces the item.

c) The next column **Bi & A** gives the probabilities $P(B_i \cap A) = P(B_i)P(A \mid B_i)$ that the item is defective and is produced by machine $\# i$. Each entry in this column equals the product of the two entries to the left. For example, $P(B_3 \cap A) = 0.25000 \times 0.02000 = 0.00500$.

d) The last column **Bi | A** gives the conditional probabilities $P(B_i \mid A) = P(B_i \cap A)/P(A)$ that the item was produced by machine $\# i$ given that the item is defective. Each entry in this column equals the entry to the left divided by the sum 0.02450 of the column entries to the left. For example, $P(B_3 \mid A) = 0.00500/0.02450 = 0.20408$ which approximates the textbook answer of $10/49$.

This table can be generated on your TI calculator. Go to the `APPS` screen and choose `Data/Matrix Editor` and select `3:New` to start a new table. From the `New` menu choose `Data` as the `Type`: and then enter your choices for the `Folder`: and `Variable`: entries. In the `c1` and `c2` columns of your Data table, enter the probabilities given in Example 2.38. Compute column `c3` as shown in the following left figure, then compute column `c4` as shown in the following right figure.





You can adjust how many digits are shown in the table cells by selecting the `Cell Width` via menu `F1` and item `9:Format`. You can type in the column titles (`A | Bi` , etc.) to identify the entries.

Bayes' Rule can be used to predict the outcome of a search scenario where we are looking for a "lost" object (sunken treasure, downed aircraft, TI calculator, etc.). Suppose the object is known to be in one of several locations that we plan to search using some strategy. Our goal is to determine the probability of eventual detection.

Simple Search Strategy

We assume that the object is known to be in one of n locations. Let A_i be the event that the object is in location i for $i = 1,2,3,\ldots,n$. Two sets of probabilities are assumed to be known. $P(A_i)$ is the probability that the object is in location i and $P(D \mid A_i)$ is the conditional probability that the object will be detected in one search of location i when the object is there. As an example, let's assume a UAV is known to have gone down while flying a mission over hostile territory. Expert opinion can determine a set of search regions and estimate the likelihood of the craft going down in each search region based on flight path, radio transmission, radar contact, etc. Experts would also estimate the likelihood of detection in each region based on the sensitivity of the search devices, topography of the region, enemy threat level in that region, etc. Our first strategy is *simple search* where all regions are searched once before probability estimates are updated based on the results of the search.

Example. Let's look at a very small, simple scenario. Assume that the UAV is known to have gone down in one of three regions. Assume the probabilities that the UAV went down in each region are estimated as:

$$P(A_1) = 0.6\,,\; P(A_2) = 0.1\,,\; P(A_3) = 0.3$$

Note: These probabilities must sum to 1 under the assumption the UAV must be in one of the three regions. Assume that the detection probabilities for each region are estimated as:

$$P(D \mid A_1) = 0.1\,,\; P(D \mid A_2) = 0.7\,,\; P(D \mid A_3) = 0.4$$

Note: These probabilities need not sum to 1. (In fact, it would be helpful if each conditional probability equaled 1!) By the Total Probability Theorem (see p. 58 of the textbook) we can compute the probability that the UAV is detected after all regions are searched once:

$$P(D) = P(A_1)P(D \mid A_1) + P(A_2)P(D \mid A_2) + P(A_3)P(D \mid A_3)$$

so

$$P(D) = (0.6)(0.1) + (0.1)(0.7) + (0.3)(0.4) = 0.25$$

With only a 25% chance of detection in the 1st search cycle, we need to determine the probabilities of detection at the end of the 2nd search cycle, 3rd cycle, etc. For this we use

Bayes' Rule (see p. 60 of the textbook) to update the object location probabilities. For example, if all regions are searched once and the UAV is not detected, then the first region object location probability is updated for the 2nd search cycle to

$$P(A_1 \mid D') = \frac{P(A_1)P(D' \mid A_1)}{P(A_1)P(D' \mid A_1) + P(A_2)P(D' \mid A_2) + P(A_3)P(D' \mid A_3)}$$
$$= \frac{(0.6)(0.9)}{(0.6)(0.9) + (0.1)(0.3) + (0.3)(0.6)} = 0.72$$

For purposes of computing this using a table (Excel or TI calculator), here is a different looking version of Bayes' Rule:

$$P(A_1 \mid D') = \frac{P(A_1 \cap D')}{P(A_1 \cap D') + P(A_2 \cap D') + P(A_3 \cap D')} = \frac{0.54}{0.75} = 0.72$$

And, in general:

$$P(A_r \mid D') = \frac{P(A_r \cap D')}{\sum_{i=1}^{n} P(A_i \cap D')} \quad \text{for } r = 1,2,3,\ldots,n$$

The following table uses the preceding formula to summarize the 1st search cycle calculations:

1st search cycle

i	D | Ai	Ai	Ai & D	Ai & not D	Ai | not D
1	0.1	0.6	0.0600	0.5400	0.7200
2	0.7	0.1	0.0700	0.0300	0.0400
3	0.4	0.3	0.1200	0.1800	0.2400
total		1.0000	0.2500	0.7500	1.0000

So, for example, the value in the upper right corner of the above table is gotten by dividing 0.5400, the entry to its left, by the sum 0.7500 of the column to its left.

Notice how the first location probability *increases* from the intial $P(A_1) = 0.6$ to the updated $P(A_1 \mid D') = 0.72$ while the second and third location probabilities *decrease* from 0.7 to 0.04 and from 0.4 to 0.24, respectively. The intuitive reason for this is: since the second and third regions have relatively high detection probabilities 0.7 and 0.4 respectively, then the fact that the UAV is not detected in the 1st cycle implies less likelihood that the UAV is in those two regions.

In the Simple Search strategy, to prepare for the 2nd search cycle we replace the column of $P(A_i)$ values with the column of $P(A_i \mid D')$ values and recompute the remaining values:

2^{nd} search cycle

i	D | Ai	Ai	Ai & D	Ai & not D	Ai | not D
1	0.1000	0.7200	0.0720	0.6480	0.8060
2	0.7000	0.0400	0.0280	0.0120	0.0149
3	0.4000	0.2400	0.0960	0.1440	0.1791
total		1.0000	0.1960	0.8040	1.0000

Again, non-detection in the 2^{nd} search cycle increases the likelihood that the UAV is in region A_1. To be summarize:

$$P(A_1) = 0.6 < P(A_1 \mid D_1') = 0.72 < P(A_1 \mid D_1' \cap D_2') = 0.806$$

where D_j is the event that the UAV is discovered during the j^{th} search cycle.

Our initial goal was to determine the probabilities of eventual detection. From the 1^{st} search cycle table we have the probability of detection $P(D_1) = 0.25$. From the 2^{nd} search cycle table — which is based on non-detection during the 1^{st} cycle — we have the conditional probability of detection $P(D_2 \mid D_1') = 0.196$. From this we can get the probability that the UAV is found on either the 1^{st} or 2^{nd} search cycle:

$$P(D_1 \cup D_2) = P(D_1) + P(D_2 \cap D_1') = P(D_1) + P(D_1')P(D_2 \mid D_1') = 0.25 + (0.75)(0.196) = 0.397$$

A more convenient way to compute this using tables is by using complements of events:

$$P(D_1' \cap D_2') = P(D_1')P(D_2' \mid D_1') = (0.75)(0.804) = 0.603$$

so (by one of DeMorgan's Laws)

$$P(D_1 \cup D_2) = 1 - P(D_1' \cap D_2') = 1 - .0603 = 0.397$$

Any way we compute this, we know that the UAV will be detected by the end of the 2^{nd} search cycle with probability 0.397. Estimations like these allow the planners to determine whether they can reach their goal in time with the allocated resources.

Once you see the pattern, we update the 2^{nd} search cycle table and use the 3^{rd} search cycle table

3^{rd} search cycle

i	D | Ai	Ai	Ai & D	Ai & not D	Ai | not D
1	0.1000	0.8060	0.0806	0.7254	0.8663
2	0.7000	0.0149	0.0104	0.0045	0.0053
3	0.4000	0.1791	0.0716	0.1075	0.1283
total		1.0000	0.1627	0.8373	1.0000

to get

$$P(D_1 \cup D_2 \cup D_3) = 1 - (0.75)(0.804)(0.8373) = 0.4951$$

So there is about a 50-50 chance that the UAV will be found by the end of the 3rd search cycle.

The following data table screen shots demonstrate one way to compute Simple Search probabilities using a TI 92 or TI Voyage 200 calculator. The example is computed using the Data/Matrix Editor application. The estimated probabilities are entered into columns `c1` and `c2`. Figures 1–4 show how to define columns `c3` through `c6`. Figure 5 shows how to define column `c2` (at least temporarily) to update the probabilities. Pressing the TI ENTER key repeatedly at this stage will generate subsequent cycle updates.

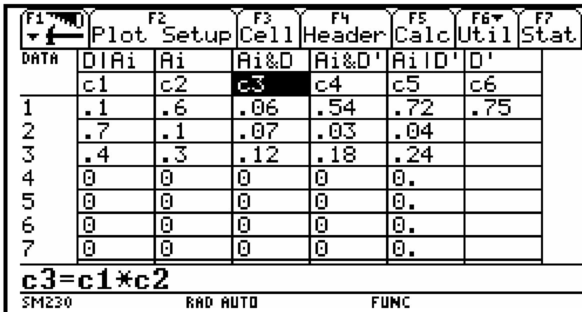


Fig. 1

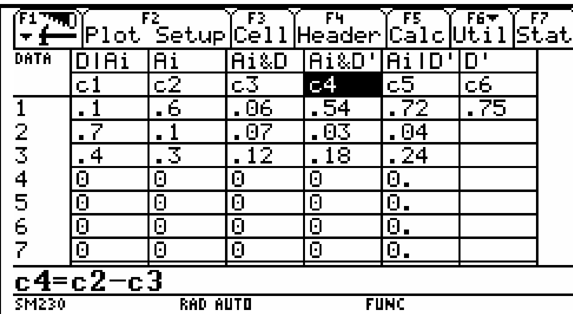


Fig. 2

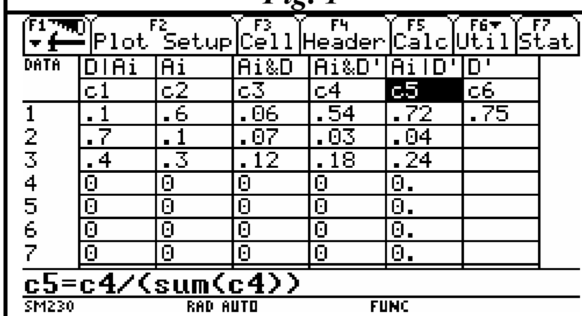


Fig. 3

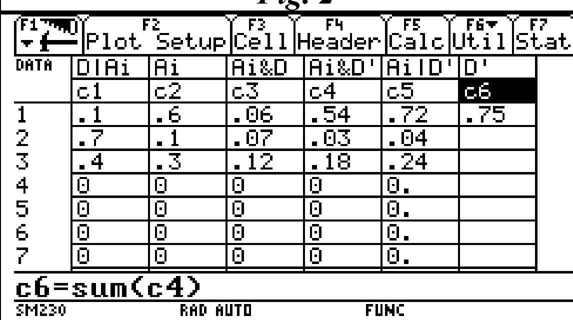


Fig. 4

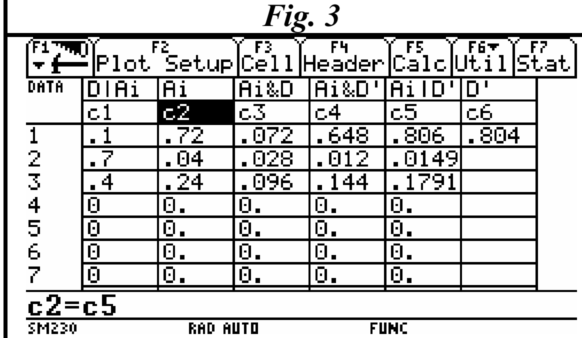


Fig. 5

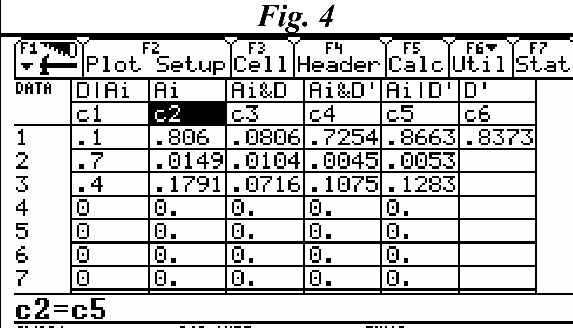


Fig. 6

Notes:

(a) The 0 entries in columns `c1` and `c2` below row 3 allow the matrix to be modified for search scenarios using a different number of search locations.

(b) Before you enter new values into column `c2` you will need to clear the `c2=c5` column definition.
(c) The titles at the top of the columns can be typed in to identify the entries.
(d) Use the `F1` menu and choose `9:Format` (or use the shortcut keystrokes: green-diamond-F) to change the cell widths.

Exercises

1. Your facility purchases large quantities of truck tires from two suppliers: A and B. Supplier A has a record of delivering tires containing 10% defectives. Supplier B has a defective rate of only 5%. Suppose 40% of your supply comes from supplier A. One tire is chosen at random from your supply.

(a) What is the probability that this tire is defective and came from supplier A?
(b) What is the probability that this tire is defective?
(c) What is the probability that this tire came from supplier A knowing that it is defective?

2. For a Simple Search scenario, part of the 1st search cycle table is given below. Fill in the missing entries.

i	D | Ai	Ai	Ai & D	Ai & not D	Ai | not D
1	0.9000	0.3000		0.0300	0.0600
2	0.3000	0.4000	0.1200	0.2800	0.5600
3	0.3000		0.0600	0.1400	
4	0.5000	0.1000	0.0500		0.1000

3. An object is known to be in one of three locations. The following table shows the results for the 1st cycle of a Simple Search scheme. Use the table to answer the following questions.

i	D | Ai	Ai	Ai & D	Ai & not D	Ai | not D
1	0.3000	0.4500	0.1350	0.3150	0.5833
2	0.7000	0.2500	0.1750	0.0750	0.1389
3	0.5000	0.3000	0.1500	0.1500	0.2778

(a) What is the conditional probability that the object will be detected if it is in the second location?
(b) What is the probability that the object will be detected in the second location at the end of this cycle?
(c) What is the probability that the object is not detected (anywhere) at the end of this cycle?
(d) What is the conditional probability that the object is in the second location knowing that it is not detected at the end of this cycle?

4. The location and detection probabilities for a Simple Search are given by the following table:

i	D | Ai	Ai
1	0.6000	0.4500
2	0.8000	0.1500
3	0.5000	0.1000
4	0.7000	0.3000

(a) Compute the probability that the object is located in region 2 and will be detected in the 1st search cycle.
(b) Compute the probability that the object is located in region 4 and will not be detected in the 1st search cycle.
(c) Compute the conditional probability that the object is in region 3 given that it was not detected in the 1st search cycle.
(d) Compute the probability that the object will be detected during the 1st search cycle.
(e) Compute the probability that the object will be detected after the 2nd search cycle.

5. Refer to the UAV example in these notes.

(a) Compute the probability $P(D_4 \mid D_1' \cap D_2' \cap D_3')$ that the UAV is detected during the 4th search cycle.
(b) Compute the probability $P(D_1 \cup D_2 \cup D_3 \cup D_4)$ that the UAV will be detected before the 5th search cycle.
(c) Determine the minimum number n of search cycles necessary so that the probability is at least 0.70 that the UAV will be detected by the end of the n^{th} search cycle.

6. Refer to the UAV example in these notes. Change the initial 1st search cycle detection probabilities to: $P(D \mid A_1) = 0.4,\ P(D \mid A_2) = 0.7,\ P(D \mid A_3) = 0.6$. Determine the minimum number n of search cycles necessary so that the probability is at least 0.90 that the UAV will be detected by the end of the n^{th} search cycle.

7. You are looking for an Iranian kilo class diesel submarine near the straits of Hormuz. Intelligence reports state that there is a 45% chance of the kilo being in the straits, a 35% chance of the kilo being in the Gulf, and a 20% chance of the kilo being in the Ocean. If the kilo is in the strait, you have a 70% chance of finding it. You have a 40% chance of finding it in the Gulf and a 10% chance if you search the ocean.

(a) You will search all 3 areas. What is your chance of finding the Iranian kilo in the Ocean?
(b) After searching all 3 areas, what is your chance of not finding the Iranian kilo?
(c) After failing to find the kilo on the initial search of all areas, what is the probability that the kilo is in the strait?
(d) If you fail to find the kilo on your second search, what is the probability the kilo is in the Gulf?

8. A TACAMO aircrew is scheduled to take off soon, but can't until they finish a FOD (Foreign Object Damage) search for the flight engineer's pen. There is a 60% chance the pen is in the cockpit, a 30% chance it is in the crew rest area, and a 10% chance it is in the communication central area. In the cockpit, there is a 20% chance of finding the pen. In the crew rest area, there is a 70% chance of finding the pen. In the communication central area, there is a 50% chance of finding the pen.

(a) What is the probability that after all three areas are searched the first time (1^{st} search cycle) the pen will be detected in the crew rest area?
(b) What is the probability that the pen is found at the end of the 1^{st} search cycle?
(c) If the pen is not found at the end of the 1^{st} search cycle, then what is the probability it is in the crew rest area?
(d) If the pen is detected at the end of the 1^{st} search cycle, then what is the probability it was in the crew rest area?
(e) Compute the probability that after all three areas are searched a second time the pen will be detected in the crew rest area.

9. The following scenario was typical of the ASW situations in the 1980's. You are the Operations officer of a P-3 squadron currently deployed to Keflavik, Iceland. The Naval Facility on Keflavik was tracking a Soviet SSBN south through the Norwegian Sea for several days, then lost it. Your squadron has been tasked to search for it. The submarine could be in one of three places: #1) west of Iceland – transiting the Greenland-Iceland gap; #2) east of Iceland – transiting the Iceland-UK gap; #3) it might have changed direction to take up a patrol station in the western portion of the Norwegian Sea.

Based on historical patterns, the submarine probably continued on through the Iceland-UK gap (75%); the intel "weenies" figure there is a 10% chance that it went west of Iceland, leaving a 15% chance that it remained in the Norwegian Sea. If it went east of Iceland, you have an 80% chance of detecting the submarine, a 60% chance of finding it west of Iceland and only a 40% of detection in the western Norwegian Sea.

You are fortunate that all of your planes are mission capable with complete crews. It is early in the month so you have plenty of sonobuoys and so you can fly in all three locations.

(a) You conduct a search of all three locations. What is the probability that you do not detect the submarine in this first search?
(b) You conduct a second search of all three locations. What is the probability that you do not detect the submarine in this second search?
(c) Assume you do not locate the submarine after three searches of all three locations. At this point, in what location is the submarine most likely to be located?