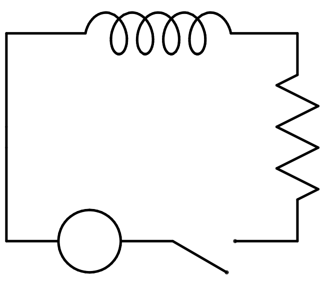
III. Example 1: R-L DC Circuit

Physical characteristics of the circuit: 60 volt DC battery connected in series with a 4 henry inductor and a 12 ohm resistor; current flows when the open switch is closed. (Note: This is Example #2 on p. 515 and #3 on p. 524 of Stewart: **Calculus—Concepts and Contexts**, 2nd ed.)



Task: Write down the Initial Value Problem associated with this circuit and solve it for the current in order to answer the following questions.

[a] Describe in words how the current changes over time.

[b] What is the current 0.1 second after the switch is closed?

[c] At what time does the current equal half the steady-state current?

[d] What is the average current over the first five time units for this circuit?

Solution: By Kirchhoff's laws we have $E_L + E_R = EMF$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the following Initial Value Problem (for $t \geq 0$):

$$4\,I'(t) + 12\,I(t) = 60, \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

We can solve for I using the method of separation of variables.

Outline of solution by *separation of variables*

First, we will divide the ODE through by 4, replace $I(t)$ by I, and use the differential notation for derivatives:

$$\frac{dI}{dt} + 3\,I = 15$$

Next, use algebra to rewrite this as

$$\frac{dI}{15 - 3\,I} = dt$$

and integrate both sides to obtain

$$-\frac{1}{3}\ln|15 - 3\,I| = t + C$$

which with the initial condition $I(0) = 0$ yields the circuit current

$$I(t) = 5 - 5\,e^{-3t}\,, \quad t \geq 0$$

More details for all these steps may be found below, after the Answers.

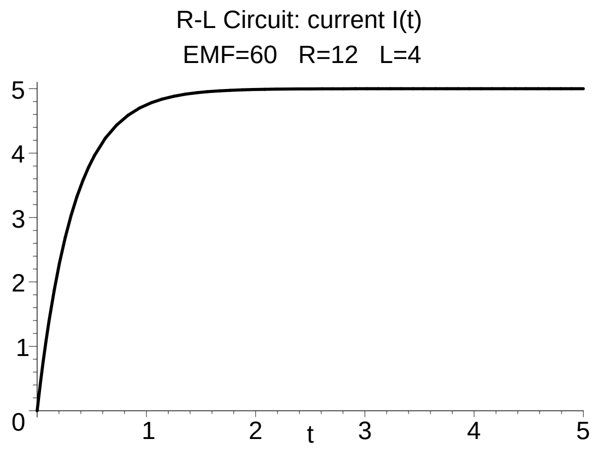
Answers:

[a] Describe in words how the current changes over time.

The following graph shows how $I(t)$ increases from 0 at $t = 0$ toward an asymptotic limit 5 as t increases:

$$\lim_{t\to\infty} I(t) = 5 - 5 \lim_{t\to\infty} e^{-3t} = 5 - 5(0) = 5$$

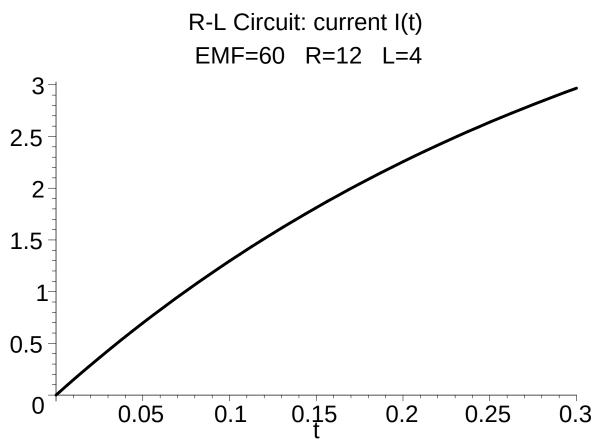
This asymptotic limit is called the *steady-state* current.



[b] What is the current 0.1 second after the switch is closed?

$$I(0.1) = 5 - 5\,e^{-3(0.1)} = 5 - 5\,e^{-0.3} \approx 1.30$$

which looks correct according to the following graph of $I(t)$.



[c] At what time does the current equal half the steady-state current?

Solve

$$I(t) = 5/2$$

or

$$5 - 5\,e^{-3\cdot t} = 2.5$$

to get

$$t = -\frac{1}{3}\ln(0.5) \approx 0.231$$

This answer could have been approximated by graphing $I(t)$ on a graphing calculator and zooming or tracing the curve. The preceding graph of $I(t)$ provides a visual check of this answer.

[d] What is the average current over the first five time units for this circuit?

By definition, the time unit for this R-L DC circuit is

$$\tau = \frac{L}{R} = \frac{4}{12} = 1/3$$

By definition of the *average value of a function* (see, e.g., p. 473 of Stewart), the average current over first five time units is

$$I_{avg} = \frac{1}{5/3}\int_0^{5/3}\left(5 - 5\,e^{-3t}\right)\,dt = 4 + e^{-5} \approx 4.01 \text{ amps}$$

Details of solution by *separation of variables*

After multiplying both sides of the ODE

$$\frac{dI}{dt} = 15 - 3\,I$$

by dt, we get the ODE in differential form

$$dI = (15 - 3\,I)\,dt$$

Divide both sides by $15 - 3\,I$ in order to *separate variables: put anything involving I on one side and anything involving t on the other side*:

$$\frac{dI}{15 - 3\,I} = dt \tag{1}$$

Now we are allowed to integrate each side separately and still have equality. The right side of equation (1) is easy:

$$\int dt = t + C$$

where C is an arbitrary constant. The left side of equation (1) looks related to the integral $\int \frac{1}{x}\,dx$. So we use the substitution

$$\begin{aligned} & x = 15 - 3\,I \\ \text{to get} \quad & \frac{dx}{dI} = -3 \\ \text{or} \quad & dI = -\frac{1}{3}dx \end{aligned}$$

Then in equation (1) we replace $15 - 3\,I$ with x and dI with $-\frac{1}{3}dx$ and integrate in order to get the left side to equal

$$\begin{aligned} \int \frac{1}{15 - 3\,I}\,dI &= \int \frac{1}{x}\left(-\frac{1}{3}dx\right) \\ &= -\frac{1}{3}\int \frac{1}{x}\,dx \\ &= -\frac{1}{3}\ln|x| + C \\ &= -\frac{1}{3}\ln|15 - 3\,I| + C \end{aligned}$$

Hence equation (1), after both sides are integrated, becomes (collecting all arbitrary constants on the right hand side as a single arbitrary constant)

$$-\frac{1}{3}\ln|15 - 3\,I| = t + C \tag{2}$$

Since there is no current when the switch is thrown, we let $I = 0$ when $t = 0$ to solve for C

$$-\frac{1}{3}\ln|15-0| = 0 + C \Longrightarrow C = -\frac{1}{3}\ln 15$$

and so equation (2) becomes

$$-\frac{1}{3}\ln|15-3\,I| = -\frac{1}{3}\ln 15 + t$$

It is usually preferable to solve for the dependent variable, I in this case. To do that, we first multiply both sides of the last equation by -3 to get

$$\ln|15-3\,I| = \ln 15 - 3t$$

then take the exponential (inverse logarithm) of both sides

$$e^{\ln|15-3\,I|} = e^{\ln 15 - 3t} \tag{3}$$

and then use a property of exponentials

$$e^{a+b} = e^a \times e^b$$

with $a = \ln 15$ and $b = -3t$ to get from equation (3)

$$\begin{aligned}|15-3\,I| &= e^{\ln 15} \times e^{-3t}\\ &= 15\,e^{-3t}\end{aligned}$$

since $e^{\ln 15} = 15$. Now $|x| = c \Longrightarrow x = \pm c$ so we have

$$15 - 3\,I = \pm 15\,e^{-3t}$$

Since we know that $I = 0$ at $t = 0$, we determine the sign to be $+$, allowing us to solve for I by dividing both sides of the last equation by 3 and then isolating I on one side

$$I(t) = 5\left(1 - e^{-3t}\right), \quad t \geq 0$$