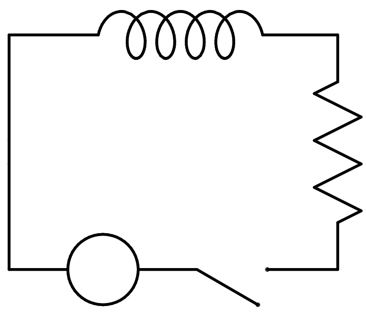
## III. Example 1: R-L Circuit with decaying EMF

Physical characteristics of the circuit: EMF $E(t) = 100e^{-t}$ connected in series with a 2 henry inductor and a 6 ohm resistor; current flows when the open switch is closed.




### Questions:

[a] Describe in words how the current changes over time.

[b] What is the current 1 second after the switch is closed?

[c] At what time does the current equal 8 amps?

[d] What is the largest current achieved and when is it achieved?

### Solution:

By Kirchhoff's laws we have $E_L + E_R = EMF$ which, with $E_L = L \cdot I'(t)$ and $E_R = R \cdot I(t)$, translates into the following Initial Value Problem (for $t \geq 0$):

$$2\,I'(t) + 6\,I(t) = 100\,e^{-t}, \qquad I(t) = 0 \quad \text{at} \quad t = 0$$

A common technique to solve DEs like this one is to introduce an *integrating factor*. After dividing through by $2$ in order to get the DE into what is called "standard form"

$$I' + 3I = 50e^{-t} \tag{$*$}$$

the left side of the DE reminds us of the product rule for derivatives

$$[f(t)g(t)]' = f'(t)g(t) + f(t)g'(t)$$

In fact, if we let $f(t) = I(t)$ and $g(t) = e^{3t}$ then we have

$$\left[Ie^{3t}\right]' = [I]'e^{3t} + I\left[e^{3t}\right]' = I'e^{3t} + 3Ie^{3t}$$

So if we multiply both sides of equation $(*)$ by the *integrating factor* $\mu = e^{\int 3\,dt} = e^{3t}$, we get

$$I'e^{3t} + 3Ie^{3t} = 50e^{-t}e^{3t}$$

or

$$\left[Ie^{3t}\right]' = 50e^{2t}$$

When we integrate both sides of the preceding equation we have

$$\begin{aligned} Ie^{3t} &= 50 \int e^{2t}\,dt \\ &= 25e^{2t} + C \end{aligned}$$

We can solve for current $I$ by multiplying through this last equation by $e^{-3t}$:

$$I(t) = 25e^{-t} + C\,e^{-3t}$$

Again, the initial condition $I(0) = 0$ allows us to determine $C$ from the last equation

$$0 = I(0) = 25e^0 + Ce^0$$

$$\Longrightarrow 0 = 25 \cdot 1 + C \cdot 1$$

$$\Longrightarrow 0 = 25 + C$$

$$\Longrightarrow C = -25$$

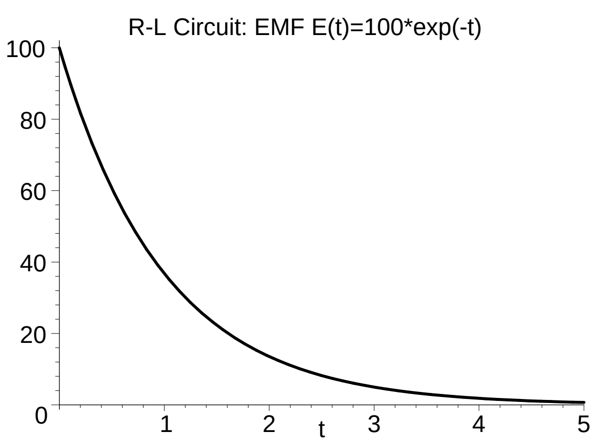
Hence for any time $t$:

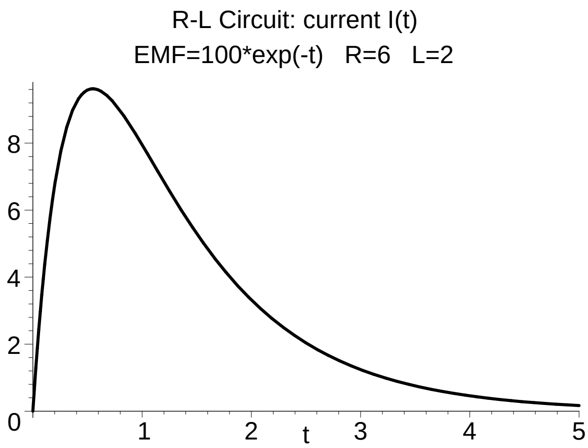
$$I(t) = 25\left(e^{-t} - e^{-3t}\right)$$

**Answers:**

[a] Describe in words how the current changes over time.

The graph below left is that of the EMF $E(t) = 100e^{-t}$ which starts out at $100$ and quickly decays toward $0$. The graph of $I$ shown below right suggests that the current starts out growing like the case where the EMF is a constant $100$ but then dies off since the EMF dies off.

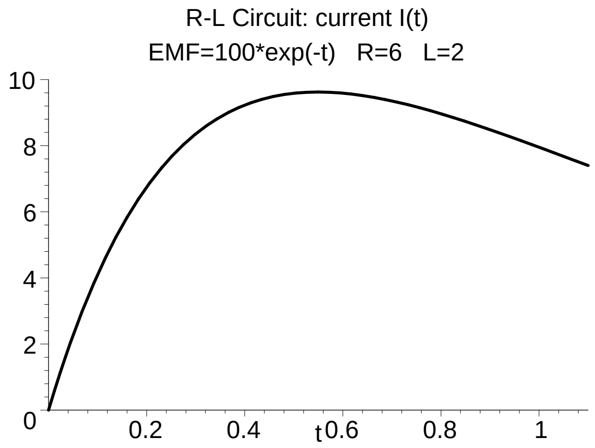



[b] What is the current 1 second after the switch is closed?

$$I(1) = 25\left(e^{-1} - e^{-3}\right) \approx 7.95$$

which agrees with the following plot of $I(t)$.




[c] At what time does the current equal 8 amps?

We can use our graphing calculator to trace the solution curve and zoom to see that it equals 8 at two times: $t \approx 0.27$ and $t \approx 0.99$. Check these with the preceding graph of $I(t)$.

[d] What is the largest current achieved and when is it achieved?

Again, using a graphing calculator we see that the maximum current $I \approx 9.62$ at $t \approx 0.55$ which agrees with the preceding graph of $I(t)$.