RIEMANN SUMS AND INTEGRALS

1. Introduction

The *integral* is one of the two fundamental concepts of Calculus. (The other is the derivative, which you studied in Calculus I.) The integral provides a way to "add up" a sum consisting of infinitely many pieces, each infinitely small. Such a sum might represent many different things (as you shall see in this course): the mass of an object, the energy needed to effect a change in a system, the volume of a solid, the length of a curve—but to begin with we focus on the integral as representing the area under a curve. This does not mean that the integral has more to do with area than any of the other concepts; area under a curve is simply the easiest situation to visualize and draw pictures of.

Goals of these notes:

(i) Be able to calculate Riemann sums for any function.

(ii) Understand the definition of the integral as a limit of Riemann sums.

(iii) Be able to calculate the integral of exponential functions as a limit of Riemann sums.

2. Riemann sums

We consider the problem of calculating the area bounded by the graph of a function $f(x)$, the x-axis, and two vertical lines $x = a$ and $x = b$ (Figure 1).

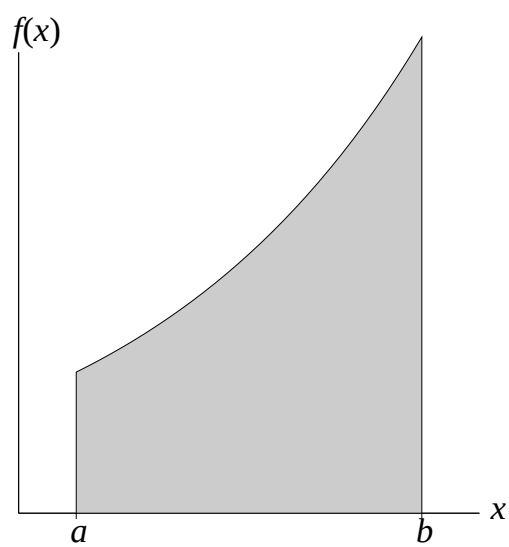

Figure 1. Area under a curve

The first step is to approximate the region as a combination of N rectangles (Figure 2). The rectangles are constructed by splitting the interval $a \leq x \leq b$ at $(N-1)$ points $x_1, x_2, x_3, \ldots, x_{N-1}$. We also define $x_0 = a$ and $x_N = b$, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b.$$

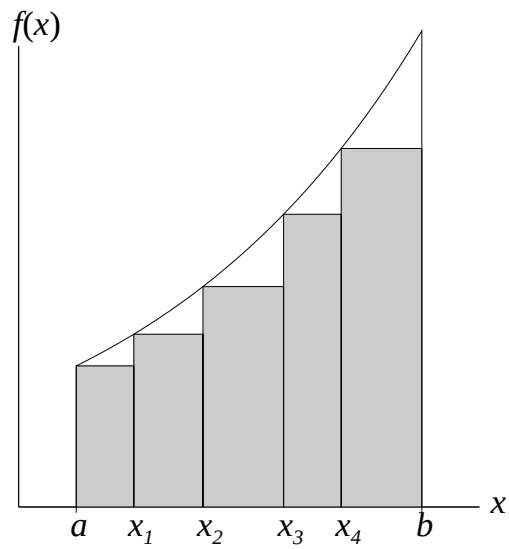

Figure 2. Area under a curve approximated by rectangles

The N rectangles have bases $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{N-1}, x_N]$. (Thus the rectangles do not overlap but fill the entire space between the lines $x = a$ and $x = b$.) The kth rectangle therefore has width Δx_k, where

$$\Delta x_k = x_k - x_{k-1}.$$

The height of the rectangle with base $[x_{k-1}, x_k]$ is set at $f(\xi_k)$, where ξ_k is any x-value between x_{k-1} and x_k. (Thus the top of the rectangle touches the graph of the function at the point $(\xi_k, f(\xi_k))$.) The kth rectangle therefore has height $f(\xi_k)$.

The area of the kth rectangle is therefore

$$\Delta x_k f(\xi_k).$$

The total area of all N rectangles is

$$\Delta x_1 f(\xi_1) + \Delta x_2 f(\xi_2) + \cdots + \Delta x_N f(\xi_N).$$

This is a *Riemann sum.*

Many choices must be made in order to calculate a Riemann sum. How do we choose the widths $\Delta x_1, \ldots \Delta x_N$? How do we choose the representative point ξ_k for each rectangle? As it turns out, these choices don't matter very much, at least not when f is bounded and piecewise-continuous and the rectangles used are very narrow—which is the case which interests us.

In order to keep things simple, we assume that all the rectangles have the same width, so that

$$\Delta x_k = \Delta x = \frac{b-a}{N};$$

and we let $\xi_k = x_{k-1}$ (the left edge of the rectangle).

Example 1: Calculate a Riemann sum for the function $f(x) = 3^x$ on the interval [2,5] using 2 equal subintervals. (This approximates the area under the graph of f between $x = 2$ and $x = 5$.)

First we determine the partition points. Since the length of the interval [2,5] is 3 and there are two subintervals, each subinterval has length $\Delta x = 1.5$. Thus:

$$\begin{aligned} x_0 &= 2, \\ x_1 &= 3.5, \\ x_2 &= 5. \end{aligned}$$

Thus the first subinterval is [2,3.5] and the second is [3.5,5].

Next we choose representative x-values ξ_1 and ξ_2 for the two subintervals. We could choose any value as long as it lies within the corresponding subinterval. The choice of value affects the answer we get, but for our ultimate goal it does not matter. For simplicity we choose the left endpoint:

$$\begin{aligned} \xi_1 &= x_0 = 2, \\ \xi_2 &= x_1 = 3.5. \end{aligned}$$

Thus the Riemann sum consists of the areas of two rectangles. Each has width $\Delta x = 1.5$. The heights of the two rectangles are $f(\xi_1) = 3^2$ and $f(\xi_2) = 3^{3.5}$. The value of the Riemann sum is therefore:

$$\begin{aligned} \Delta x f(\xi_1) + \Delta x f(\xi_2) &= \Delta x (f(\xi_1) + f(\xi_2)) \\ &= (1.5)\left(3^2 + 3^{3.5}\right) \\ &= 83.6481.... \end{aligned}$$

Example 2: Calculate a Riemann sum for the function $f(x) = 3^x$ on the interval [2,5] using 5 equal subintervals. (This approximates the area under the graph of f between $x = 2$ and $x = 5$. With smaller subintervals [narrower rectangles], we expect the approximation to improve.)

First we determine the partition points. Since the length of the interval [2,5] is 3 and there are five subintervals, each subinterval has length $\Delta x = 0.6$. Thus:

$$\begin{aligned} x_0 &= 2, \\ x_1 &= 2.6, \\ x_2 &= 3.2, \\ x_3 &= 3.8, \\ x_4 &= 4.4, \\ x_5 &= 5. \end{aligned}$$

Thus the subintervals are [2,2.6], [2.6,3.2], [3.2,3.8], [3.8,4.4], and [4.4,5].

Next we choose representative x-values $\xi_1 \dots \xi_5$ for the five subintervals. Again, we could choose any value as long as it lies within the corresponding subinterval, but for simplicity we choose the left endpoint:

$$\begin{aligned}
\xi_1 &= x_0 = 2, \\
\xi_2 &= x_1 = 2.6, \\
\xi_3 &= x_2 = 3.2, \\
\xi_4 &= x_3 = 3.8, \\
\xi_5 &= x_4 = 4.4.
\end{aligned}$$

Thus the Riemann sum consists of the areas of five rectangles. Each has width $\Delta x = 0.6$. The heights of the five rectangles are $f(\xi_1) = 3^2$, $f(\xi_2) = 3^{2.6}$, $f(\xi_3) = 3^{3.2}$, $f(\xi_4) = 3^{3.8}$, and $f(\xi_5) = 3^{4.4}$. The value of the Riemann sum is therefore:

$$\begin{aligned}
\Delta x f(\xi_1) + &\Delta x f(\xi_2) + \Delta x f(\xi_3) + \Delta x f(\xi_4) + \Delta x f(\xi_5) \\
&= \Delta x (f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4) + f(\xi_5)) \\
&= (0.6)\left(3^2 + 3^{2.6} + 3^{3.2} + 3^{3.8} + 3^{4.4}\right) \\
&= 150.453....
\end{aligned}$$

Example 3: Calculate a Riemann sum for the function $f(x) = 3^x$ on the interval [2,5] using N equal subintervals. (This requires that we generalize the previous two examples to an unknown value of N. We expect that this approximates the area under the curve better and better for larger and larger values of N.)

First we determine the partition points. Since the length of the interval [2,5] is 3 and there are N subintervals, each subinterval has length $\Delta x = 3/N$. Note that the values of N and Δx are linked. We can determine Δx from N or vice-versa. All our computations could be done in terms of N, or in terms of Δx. It turns out that Δx is more convenient. So we have:

$$\begin{aligned}
x_0 &= 2, \\
x_1 &= 2 + \Delta x, \\
x_2 &= 2 + 2\Delta x, \\
&\dots \\
x_{N-1} &= 2 + (N-1)\Delta x, \\
x_N &= 2 + N\Delta x = 5.
\end{aligned}$$

Thus the subintervals are $[2, 2+\Delta x]$, $[2+\Delta x, 2+2\Delta x]$, ..., $[2+(N-1)\Delta x, 2+N\Delta x]$.

Next we choose representative x-values $\xi_1 \dots \xi_k$ for the $k = 1 \dots N$. As before we choose $\xi_k = x_{k-1}$.

Thus the Riemann sum consists of the areas of N rectangles. Each has width Δx. The heights of the N rectangles are $f(\xi_1), \dots f(\xi_N)$. The value of the Riemann sum is therefore:

$$\begin{aligned}
\Delta x f(\xi_1) + \dots + \Delta x f(\xi_N) &= \Delta x (f(\xi_1) + \dots + f(\xi_N)) \\
&= \Delta x \left(3^2 + 3^{2+\Delta x} + 3^{2+2\Delta x} + \dots + 3^{2+(N-1)\Delta x}\right).
\end{aligned}$$

We can simplify this by recognizing that the sum in parentheses is a geometric series. Recall that the sum of a finite geometric series is given by:

$$a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a - ar^k}{1 - r}.$$

In our Riemann sum, we have $a = 3^2$ (the first term), $r = 3^{\Delta x}$ (the ratio of two successive terms) and $k = N$ (the number of terms). So the Riemann sum equals:

$$\Delta x \left(\frac{3^2 - 3^{2+N\Delta x}}{1 - 3^{\Delta x}} \right).$$

We have one more simplifying step: replace $N\Delta x$ by 3 (recall how we obtained Δx):

$$\Delta x \left(\frac{3^2 - 3^{2+3}}{1 - 3^{\Delta x}} \right) = -\frac{234\Delta x}{1 - 3^{\Delta x}} = L(\Delta x).$$

Once we have figured this out, it is simple to get better and better approximations to the area by using smaller and smaller values of Δx:

$$\begin{aligned} L(0.6) &= 150.453, \\ L(0.1) &= 201.51, \\ L(0.01) &= 211.828. \end{aligned}$$

3. Integrals

Let $L_{f,a,b}(\Delta x)$ denote the Riemann sum for a function f on an interval $[a, b]$, using subintervals of width Δx and using the left endpoint x_{k-1} of each subinterval as the representative point ξ_k—just as we did in the previous section. If f is bounded and piecewise-continuous, then it can be proven that the limit

$$\lim_{\Delta x \to 0} L_{f,a,b}(\Delta x)$$

exists. This is the definition of the *integral* of f on $[a, b]$:

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} L_{f,a,b}(\Delta x).$$

This is also called the *Riemann integral*. It gives us the exact area under the graph of f between $x = a$ and $x = b$.

(This definition is specific enough for our purposes, but in fact a much stronger statement is possible. For a bounded piecewise-continuous function the subintervals can have varying widths Δx_k and the representative point ξ_k can be chosen anywhere within its subinterval and it can still be proven that the limit [as the maximum Δx_k goes to zero] exists and always takes the same value—which is the value of the integral.)

Example 4: Find

$$\int_2^5 3^x \, dx$$

using Riemann sums. (This will tell us *exactly* the area under the curve.)

Recall that in Example 3 we found the value $L_{3^x,2,5}(\Delta x)$ of the Riemann sum for an arbitrary value of N:

$$L_{3^x,2,5}(\Delta x) = -\frac{234\Delta x}{1-3^{\Delta x}}.$$

The rest of the problem is fairly straightfoward. All we need do is take the limit as $N \to \infty$ (or, equivalently, $\Delta x \to 0$):

$$\int_2^5 3^x \, dx = \lim_{\Delta x \to 0} -\frac{234\Delta x}{1-3^{\Delta x}}.$$

Straight substitution of 0 for Δx gives us $0/0$. So we use L'Hôpital's Rule (separately differentiate the top and bottom of the fraction), and obtain:

$$\lim_{\Delta x \to 0} -\frac{234\Delta x}{1-3^{\Delta x}} = \lim_{\Delta x \to 0} -\frac{234}{-3^{\Delta x}\ln 3} = \frac{234}{\ln 3}.$$

Thus

$$\int_2^5 3^x \, dx = \frac{234}{\ln 3} = 212.996...,$$

and this is exactly the area under the graph of 3^x from $x = 2$ to $x = 5$.

4. Exercises

1-4: Calculate the Riemann sum for the given function on the given interval, with the given number N of subrectangles. Sketch the graph of f on the given interval along with the rectangles used in your calculation.

1. $f(x) = 4^x$ on $[0, 3]$, $N = 3$.
2. $f(x) = (1/2)^x$ on $[1, 3]$, $N = 4$.
3. $f(x) = 2^x$ on $[2, 6]$, $N = 2$.
4. $f(x) = 3^x$ on $[-1, 1]$, $N = 4$.

5-8: (a) Calculate the Riemann sum for the given function on the given interval, for an arbitrary number N of subrectangles. (b) Express your answer in terms of the width Δx of each subrectangles.

5. $f(x) = 4^x$ on $[0, 3]$.
6. $f(x) = (0.5)^x$ on $[1, 3]$.
7. $f(x) = 2^x$ on $[2, 6]$.
8. $f(x) = 3^x$ on $[-1, 1]$.

9-12: Use Riemann sums to calculate the integral.

9. $\int_0^3 4^x \, dx$.
10. $\int_1^3 (0.5)^x \, dx$.
11. $\int_2^6 2^x \, dx$.
12. $\int_{-1}^1 3^x \, dx$.

13. (a) Find a Riemann sum for the function $f(x) = \cos x$ on the interval $[0, .8]$ with four subintervals.

(b) Use the formula

$$\cos(0\alpha) + \cos(1\alpha) + \cos(2\alpha) + \cdots + \cos([N-1]\alpha) = \frac{1 - \cos(N\alpha)}{2} + \frac{\sin(N\alpha)}{2\tan(\alpha/2)}$$

to find a formula for the value of the Riemann sum for $f(x) = \cos x$ on the interval $[0, 0.8]$ with an arbitrary number N of subintervals.

(c) Use your answer from part (b) to find

$$\int_0^{0.8} \cos x \, dx.$$

14. For the function $f(x) = 3^x$ on the interval $[0, 1]$:

(a) Calculate the Riemann sum using 5 subintervals and the left endpoint of each subinterval as the representative point.

(b) Calculate the Riemann sum using 5 subintervals and the *right* endpoint of each subinterval as the representative point.

(c) What is the difference between your answers for parts (a) and (b)?

(d) Sketch on a single set of axes the graph of f and the rectangles used in both parts (a) and (b). Can you explain the value of the difference by looking at the diagram?

(e) Find a formula for the difference between the Riemann sums calculated with left and right endpoints for an arbitrary number N of subintervals.

(f) What is the limit of your answer from part (e) as $N \to \infty$?

5. Answers

1. 21

2. 0.640165

3. 40

4. 1.82137

5. $-\frac{189}{N(1-4^{3/N})} = -\frac{63\Delta x}{1-4^{\Delta x}}$

6. $\frac{0.75}{N(1-(0.5)^{2/N})} = \frac{0.375\Delta x}{1-(0.5)^{\Delta x}}$.

7. $-\frac{240}{N(1-2^{4/N})} = -\frac{60\Delta x}{1-2^{\Delta x}}$

8. $-\frac{16}{3N(1-3^{2/N})} = -\frac{8\Delta x}{3(1-3^{\Delta x})}$

9. $63/\ln 4 \approx 43.4449$

10. $3/(8\ln 2) \approx 0.541011$

11. $60/\ln 2 \approx 86.5617$

12. $8/(3\ln 3) \approx 2.4273$

13. (a) 0.745293

(b) $\frac{0.4(1-\cos(0.8))}{N} + \frac{0.4\sin(0.8)}{N\tan(0.4/N)}$

(c) $\sin(0.8) \approx 0.717356$