MATHEMATICS PROBLEM #117

What is the largest number of regions $r(n)$ that a plane is divided into by n straight lines in the plane? Give $r(n)$ as a function of n and explain why your answer is correct.

Advanced Problem: What is the largest number of regions $r(n, d)$ that d-dimensional Euclidean space is divided into by n hyperplanes? Give
$r(n, d)$ as a function of n and d, and find formulas $b(n, d)$, the number of regions that are bounded, and $u(n, d)$, the number of regions that are unbounded. Of course, $r(n, d) = b(n, d) + u(n, d)$. Explain why these numbers are correct.

Each midshipman submitting a correct solution to Problem 117 by noon on Sunday 1 April 2001 will win a cookie, or two cookies for the regular problem and the advanced problem in the case $d = 3$, or three cookies for the advanced problem in its general form. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his mailbox in Chauvenet 301.

Correct solutions to Mathematics Problem #116 were submitted by Midshipmen Aron T. Foster, Aaron Geary, and Andrew J. Privette. Midn. Geary's solution is posted on the board. Professor Mark Kidwell submitted a correct solution to Advanced Prob. 116.

Solution to Advanced Mathematics Problem 116

Consider the ring $\mathbf{Z}_p$ of integers modulo the prime number p. This ring is the finite field $\mathbf{F}_p = \{0, 1, \ldots, p-1\}$ An integer is divisible by p if and only if it has remainder 0 when divided by p. To find if a matrix with integer entries is divisible by p, we can reduce its entries modulo p (That is, replace each of its entries with its remainder when it is divided by p.) and calculate the determinant of the resulting matrix modulo p. The determinant of the original matrix is divisible by p if and only if the determinant of the reduced matrix is 0 modulo p. By a theorem of linear algebra, a matrix with entries in a field has determinant 0 if and only if it is not invertible if and only if its rows are dependent. Since any one of the entries 0, 1, … , p-1 are equally likely, any one of the p^m $(m = n^2)$ n x n matrices with elements chosen from $\mathbf{F}_p$ is equally likely. If I is the number of invertible matrices, then the probability that a matrix has determinant divisible by p is $Pr = (p^m - I)/ p^m$. Let us calculate the number I of invertible n x n matrices with entries chosen from $\mathbf{F}_p$. Such a matrix is invertible if and only if its rows are linearly independent. The first row can be anything except the row of all zeros, so there are $p^n - 1$ possible choices of the first row. Given any such choice for the first row, the second row can be any one of the p^n possible rows except it cannot be one of the p multiples of the first row, so there are $p^n - p$ choices for the second row. The third row must exclude the p^2 multiples of the first two rows, so there are $p^n - p^2$ possibilities for the third row. Similarly, there are $p^n - p^{k-1}$ choices for the kth row. Thus $I = (p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})$, and the probability that the determinant of the matrix is divisible by p is

$$Pr = 1 - I/p^m = 1 - (1- p^{-n})(1 - p^{1-n}) \cdots (1 - p^{-1}).$$

Remark: For $n = p = 2$, $Pr = 1 - (1-2^{-2})(1 - 2^{-1}) = 1 - (3/4)(1/2) = 1 - 3/8 = 5/8$, which is the probability that the determinant of a 2 x 2 matrix with random integer entries is even.