MATHEMATICS PROBLEM 124

Three numbers, *a, b,* and *c* are each chosen randomly and independently from the unit interval *[0, 1]* using the uniform distribution. What is the probability that a triangle can be formed with sides of length *a, b,* and *c* ?

Each midshipman submitting a correct solution with a correct explanation to Problem 124 by 1700 on Tuesday 30 April 2002 will win a cookie. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

No correct solutions to Mathematics Problem 123 were submitted. My solution to Mathematics Problem 123 is posted on the board and is on the reverse side of this sheet.

MATHEMATICS PROBLEM 123

Two long (length at least $a + b + c$ units) horizontal halls meet at right angles. The east-west hall is a units wide, and the north-south hall is b units wide. Both have vertical walls and a horizontal ceiling c units high. What is the length of the longest rigid rod (of negligible diameter) which can be carried along the east-west hall, around the corner, and along the north-south hall?

Solution

First we find the length L_H of the longest rigid rod that can go through the halls in a horizontal position. To find this length, let L(°) be the length of the horizontal line segment through the corner of the halls which touches the opposite walls of the two halls and makes an angle of ° with the east- west wall. Then L(°) = A + B, where A is that portion of the line segment crossing the east-west hall of width a, and B is that portion of the line segment crossing the north-south hall of width b. Then a/A = sin(°), since A is the hypotenuse of the right triangle with angle ° and opposite side a, and b/B = cos(°), since B is the hypotenuse of the right triangle with angle ° and adjacent side b. Hence,

$$L(^\circ) = A + B = a \csc(^\circ) + b \sec(^\circ),$$

is minimized when $L^{\circ}(^\circ) = -a \csc(^\circ)\cot(^\circ) + b \sec(^\circ)\tan(^\circ) = 0$, or when ° = ° is the angle such that

$$a/b = (\sec(^\circ)\tan(^\circ))/(\csc(^\circ)\cot(^\circ)) = \tan^3(^\circ).$$

Hence, the length L_H of the longest rigid rod that can go through the halls in a horizontal position is the minimum value of L(°), and so

$$L_H = L(^\circ) = a \csc(^\circ) + b \sec(^\circ) = a(1+\cot^2(^\circ))^{1/2} + b(1+\tan^2(^\circ))^{1/2}$$

$$= a(1+ (b/a)^{2/3})^{1/2} + b(1+(a/b)^{2/3})^{1/2} = (a^2+ a^{4/3}b^{2/3})^{1/2} + (b^2+ a^{2/3}b^{4/3})^{1/2} .$$

Now if we lay this length L_H on the floor of the hall and connect one end of L_H to the point on the ceiling directly above the other end, we see that the hypotenuse of this right triangle with sides L_H and c has length

$$L_{MAX} = (L_H{}^2 + c^2)^{1/2}$$

$$= [a^2+ a^{4/3}b^{2/3} + 2(2 a^2b^2+ a^{8/3}b^{4/3} + a^{4/3}b^{8/3})^{1/2} + b^2+ a^{2/3}b^{4/3} + c^2]^{1/2} ,$$

which is the length of the longest rigid rod that can be carried through the hallways.