# MATHEMATICS PROBLEM 124

Three numbers, $a$, $b$, and $c$ are each chosen randomly and independently from the unit interval $[0, 1]$ using the uniform distribution. What is the probability that a triangle can be formed with sides of length $a$, $b$, and $c$ ?

**Solution.** We can consider $a$ to lie in the interval $[0, 1]$ on the $x$-axis, $b$ to lie in the interval $[0, 1]$ on the $y$-axis, and $c$ to lie in the interval $[0, 1]$ on the z-axis. Thus choosing $a$, $b$, and $c$ is equivalent to randomly choosing a point in the unit cube

$\boldsymbol{B} = [0, 1] \times [0, 1] \times [0, 1] = \{(x, y, z) \text{ in } \boldsymbol{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$

in the first octant of $\boldsymbol{R}^3$. A triangle can be formed with such sides of length $a$, $b$, and $c$ if and only if $0 < a \leq 1$, $0 < b \leq 1$, $0 < c \leq 1$, $a < b + c$, $b < a + c$, and $c < a + b$. Now it can be seen that the point $P = (a, b, c)$ in $\boldsymbol{B}$ satisfies these inequalities if it does not lie in any of the three tetrahedra

$$\boldsymbol{T}_1 = \{(x, y, z) \text{ in } \boldsymbol{B} : 0 \leq x - y - z\}, \quad \boldsymbol{T}_2 = \{(x, y, z) \text{ in } \boldsymbol{B} : 0 \leq -x + y - z\},$$

and

$$\boldsymbol{T}_3 = \{(x, y, z) \text{ in } \boldsymbol{B} : 0 \leq -x - y + z\}.$$

( $\boldsymbol{T}_1$ has vertices $(0, 0, 0), (1, 0, 0), (1, 1, 0)$, and $(1, 0, 1)$, $\boldsymbol{T}_2$ has vertices $(0, 0, 0), (0, 1, 0), (1, 1, 0)$, and $(0, 1, 1)$, and $\boldsymbol{T}_3$ has vertices $(0, 0, 0), (0, 0, 1), (1, 0, 1)$, and $(0, 1, 1)$.) Each of these tetrahedra has volume 1/6, and the volume of the intersection of any two of the tetrahedra is 0.

Now we see that a triangle can be formed from the numbers $a, b, c$ chosen from the interval $[0, 1]$ if and only if the point $P = (a, b, c)$ lies in the region $\boldsymbol{S} = \boldsymbol{B} - \boldsymbol{T}_1 - \boldsymbol{T}_2 - \boldsymbol{T}_3$ obtained by deleting the three tetrahedra from the cube $\boldsymbol{B}$. Thus the probability that a triangle can be formed with sides of lengths $a$, $b$, and $c$ chosen from the interval $[0, 1]$ is the volume of $\boldsymbol{S}$,

$\text{Vol}(\boldsymbol{S}) = \text{Vol}(\boldsymbol{B}) - \text{Vol}(\boldsymbol{T}_1) - \text{Vol}(\boldsymbol{T}_2) - \text{Vol}(\boldsymbol{T}_3) = 1 - 1/6 - 1/6 - 1/6 = 1/2$ .

# MATHEMATICS PROBLEM 125

Let A and B be fixed points in three dimensional space. Find and describe the locus of all points P that are twice as far from A as from B. (That is, $|AP| = 2|BP|$, where $|RS|$ denotes the distance between the points R and S.)

Each midshipman submitting a correct solution with a correct explanation to Problem 125 by 1700 Monday 30 September 2002 will win a cookie. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his mailbox in Chauvenet 301.

No correct solutions to Mathematics Problem 124 were submitted. My solution to Mathematics Problem 124 is posted on the board and is on the reverse side of this sheet.