MATHEMATICS PROBLEM 126

Find the value of the following infinite continued fraction, assuming that it converges:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}} \quad \cdots \quad \cdots \quad .$$

(For two more cookies: Interpret the above as an infinite sequence and show that it converges.)

Each midshipman submitting a correct solution with a correct explanation to Problem 126 by 1700 Thursday 31 October 2002 will win a cookie. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Mathematics Problem 125 were submitted by Midn Alan W. Kruppa and by LCDR David Ruth, USNA '91. My solution to Mathematics Problem 125 is posted on the board and is on the reverse side of this sheet.

MATHEMATICS PROBLEM 125

Let A and B be fixed points in three dimensional space. Find and describe the locus of all points P that are twice as far from A as from B. (That is, $|AP| = 2|BP|$, where $|RS|$ denotes the distance between the points R and S.)

Solution. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{r}$ be position vectors for the points A, B, and C, respectively. Then

$$|AP| = 2|BP|$$

if and only if

$$|\mathbf{r} - \mathbf{a}| = 2|\mathbf{r} - \mathbf{b}|$$

if and only if

$$|\mathbf{r} - \mathbf{a}|^2 = 4|\mathbf{r} - \mathbf{b}|^2$$

if and only if

$$(\mathbf{r} - \mathbf{a})^2 = 4(\mathbf{r} - \mathbf{b})^2$$

(Note that if $\mathbf{u}$ is a vector, then $\mathbf{u}^2 = \mathbf{u}\circ\mathbf{u}$ is the dot product of $\mathbf{u}$ with itself.) if and only if

$$\mathbf{r}^2 - 2\mathbf{a}\circ\mathbf{r} + \mathbf{a}^2 = 4(\mathbf{r}^2 - 2\mathbf{b}\circ\mathbf{r} + \mathbf{b}^2)$$

if and only if

$$3\mathbf{r}^2 - 2(4\mathbf{b} - \mathbf{a})\circ\mathbf{r} + 4\mathbf{b}^2 - \mathbf{a}^2 = 0$$

if and only if

$$\mathbf{r}^2 - (2/3)(4\mathbf{b} - \mathbf{a})\circ\mathbf{r} = (1/3)(\mathbf{a}^2 - 4\mathbf{b}^2)$$

if and only if

$$\mathbf{r}^2 - (2/3)(4\mathbf{b} - \mathbf{a})\circ\mathbf{r} + [(1/3)(4\mathbf{b} - \mathbf{a})]^2 = (1/3)(\mathbf{a}^2 - 4\mathbf{b}^2) + [(1/3)(4\mathbf{b} - \mathbf{a})]^2$$

if and only if

$$[\mathbf{r} - (1/3)(4\mathbf{b} - \mathbf{a})]^2 = (1/9)[3(\mathbf{a}^2 - 4\mathbf{b}^2) + 16\mathbf{b}^2 - 8\mathbf{a}\circ\mathbf{b} + \mathbf{a}^2]$$

if and only if

$$[\mathbf{r} - (1/3)(4\mathbf{b} - \mathbf{a})]^2 = (4/9)(\mathbf{a}^2 - 2\mathbf{a}\circ\mathbf{b} + 4\mathbf{b}^2)$$

if and only if

$$(*) \quad (\mathbf{r} - \mathbf{c})^2 = [(2/3)(\mathbf{b} - \mathbf{a})]^2 = d^2$$

where $\mathbf{c} = (4/3)\mathbf{b} - (1/3)\mathbf{a}$ is the position vector of the point C on the line joining A and B which is "4/3 of the way from A towards B" (That is, C lies at a distance $(1/3)|AB|$ from B on the side away from A.) , and $d = (2/3)|AB|$ is two thirds of the distance from A to B. The equation (*) is the equation of a sphere with center C and radius d, the locus of all points P which are twice as far from A as from B.