MATHEMATICS PROBLEM 129

a. How many sequences of ten positive integers have a sum less than or equal to a thousand?

b. How many sequences of ten nonnegative integers have a sum less than or equal to a thousand?

(For example, 210 sequences of four positive integers have a sum less than or equal to ten. Three examples of such sequences are (1,2,5,2), (2,2,5,1), and (1,2,3,4).)

Hint: Similar to Math Prob 128 – see solution on back!

Each midshipman submitting a correct solution with a correct explanation to part a or to part b of Problem 129 by 1700 Friday 28 February 2003 will win a cookie. (Two cookies for both a and b!) Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

A correct solution to Mathematics Problem 128 was submitted by Prof. Craig Bailey. His solution is posted on the board. The problem is a classical combinatorics question and many solutions appear in the literature. My solution is on the back and on the board.

Mathematics Problem 128

a. How many sequences of ten positive integers sum to a thousand?
b. How many sequences of ten nonnegative integers sum to a thousand?

(For example, 84 sequences of four positive integers sum to ten. Three examples of such sequences are (1,2,5,2), (2,2,5,1), and (1,2,3,4).)

Solution. a. Consider the general problem: How many sequences of positive integers have sum ? This is clearly the number of ways we can put balls (or counters) in boxes while leaving no box empty. If we line all balls up in a single row, we can choose how many balls go into each of the boxes by laying sticks in spaces chosen among the spaces between the balls. (For , the sequences (1,2,5,2), (2,2,5,1), and (1, 2,3,4) are given by “ball and stick” diagrams °|°°|°°°°°|°° , °°|°°|°°°°°|° , °|°°|°°°|°°°° , respectively.) There are “ choose ” ways to do this. Thus there are C(s-1,k-1) = (s-1)! /((s-k)!(k-1)!) sequences of k positive integers which sum to s. Thus, there are C(9,3) = 9x8x7/(3x2x1) = 84 sequences of four positive integers which sum to ten and C(999, 9) = 999x998x…x991/(9x8x…x1) = 2634095604619702128324 sequences of ten positive integers which sum to 1000.

b. Consider the general problem: How many sequences of nonnegative integers have sum ? We can convert this problem to the positive integer problem in part a by taking and replacing by . Hence thare are C(s+k-1,k-1) = (s+k-1)!/((s)!(k-1)!) sequences of k nonnegative integers which sum to s. Thus there are C(1009,9) = 1009x1008x…x1001/(9x8x…x1) = 2882163562453289940826 sequences of ten nonnegative integers which sum to 1000.