MATHEMATICS PROBLEM 132

Construct two 2 x 2 matrices A and B by flipping a fair coin to determine each entry; the entry is a 1 if the coin comes up heads and a 0 if the coin comes up tails. What is the probability that every entry in the matrix C = AB – I is even? (I is the 2 x 2 identity matrix, with two 1's down the diagonal and two 0's in the other positions.)

Each midshipman submitting a correct solution with a correct explanation to Problem 132 by 1700 Wednesday 1 October 2003 will be recognized on next month's problem sheet. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his mailbox in Chauvenet 301.

No one submitted a correct solution to Mathematics Problem 131. My solution is given below and on the display board.

MATHEMATICS PROBLEM 131.

Find all real valued functions f such that

$$\int f(g(x))dx = f(\int g(x)dx)$$

for every real valued function g. (This problem was inspired by a student who incorrectly wrote " $\int \sin(x)dx = \sin(x^2/2)$. ")

Solution. The statement of the problem would probably have been improved by requiring both f and g to be continuous. And the indefinite integral is a bit, well, indefinite, so it may have been better to replace each Int { ¿ with Int rSub { size 8{0} } rSup { size 8{t} } {} . That may explain why there were no solutions submitted! However, we will solve the problem as stated, with the assumption that both f and g are continuous:

First we observe that if a is any constant real number and f is defined by

$$f(x) = ax$$

for each real number x, then

$\int f(g(x))dx = \int ag(x)dx = a\int g(x)dx = f(\int g(x)dx)$, so f satisfies the given condition.

But are there any other functions f that work? It follows from the equality

$$(1) \qquad \int f(g(x))dx = f(\int g(x)dx)$$

of indefinite integrals that

$$(2) \quad \int_0^t f(g(x))dx = \left(\int f(g(x))\,dx\right)\Big|_0^t = \left(f\left(\int g(x)dx\right)\right)\Big|_0^t .$$

Since f must work for any g, we can pick a particular function g designed to give us information about f. For example, let g be the constant function given by $g(x)=0$. Hence, $g(x)=0$ gives us $f(0)t=$ $\int_0^t f(0)dx = \left(f\left(\int 0\,dx\right)\right)\Big|_0^t = f(c)-f(c)=0$, where c is the constant of integration, $c=$ $\int 0\,dx$, and so $f(0)=0$. Now, taking $g(x)=1$ for all x in (2) gives us $f(1)t=$ $\int_0^t f(1)dx = \left(f\left(\int dx\right)\right)\Big|_0^t = \left(f(x+d)\right)\Big|_0^t$ $=f(t+d)-f(d)$, where d is the constant of integration in $\int dx = x+d$. Taking a to be the constant $f(1)$, we have $f(t+d)=at+f(d)$, so $0=f(0)=f(-d+d)$ $=-ad+f(d)$ implies that $f(d)=ad$ and so $f(t+d)=a(t+d)$. Replacing $t+d$ by x shows that f must be the function given by

$$f(x)=ax \quad .$$