MATHEMATICS PROBLEM 139

a. What is the volume of a regular tetrahedron with side length 1 ?

b. What is the volume of a regular octahedron with side length 1 ?

Each midshipman submitting a correct solution with a correct explanation to Problem 139 by 1700 Friday 30 April 2004 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his mailbox in Chauvenet 301.

Midshipmen Gabriel Campion, Jordan Kehrer and Stephan Koev, and CDR Vincent van Joolen each submitted a correct solution to Mathematics Problem 138. A solution is on the back of this page and is posted on the right wall at the Bancroft entrance to Chauvenet Hall.

Remark: There was an error in the solution to Mathematics Problem 137. The matrix A should have had last row [1 1 0 0] instead of [1 0 0 0] as stated.

MATHEMATICS PROBLEM 138

Describe and give an equation for the locus of all those points (x, y, z) given by the parametric equations

$$x = 3u + v + 4,$$
$$y = u + 5v + 9,$$
$$z = 2u + 6v + 5,$$

as the parameters u and v assume all real values.

SOLUTION

The equations

$$(1) \quad x = 3u + v + 4,$$
$$(2) \quad y = u + 5v + 9,$$
$$(3) \quad z = 2u + 6v + 5,$$

imply

$$(4) \quad x - 3y = -14v - 23$$

and

$$(5) \quad z - 2y = -4v - 13.$$

Taking twice (4) and subtracting seven times (5) yields the equation

$$(6) \quad 2x + 8y - 7z = 45.$$

This shows that any real numbers u and v, the numbers x, y, z given by equations (1), (2), and (3) satisfy the equation (6), and so the locus lies on the plane given by (6).

It remains to show that every point on the plane given by (6) is obtained from the equations (1), (2), and (3) for some choice of u and v. To see this, suppose that (x, y, z) is any point on the plane (6). Then it follows from (1) and (2) that $y - 5x = -14u - 11$, and that $x - 3y = -14v - 23$, and hence, that

$$(7) \quad u = (5x - y + 11)/14$$

and

$$(8) \quad v = (-x + 3y - 23)/14$$

are values of u and v that give the point (x, y, z) on the plane (6). This completes the proof that the plane given by (6) is the locus of points (x, y, z) given by equations (1), (2), and (3).

Remark: To see the need for the last paragraph of the solution, observe that every point (x, y, z) given by the parametric equations $x = u - v + 1$, $y = -u + v + 1$, and $z = 2u - 2v + 3$ satisfies the equation $3x + y + z = 7$, but not every point on the plane is given by the parametric equations. For example, $(1, 2, 2)$ lies on the plane but is not given by the parametric equations. The locus of the parametric equations is the line $x = t + 1$, $y = -t + 1$, and $z = 2t + 3$.