MATHEMATICS PROBLEM 144

Basketball star Shanille O'Keal's team statistician keeps track of the number, S(N), of successful free throws she has made in her first N attempts of the season. Early in the season, S(N) was less than 80% of N, but by the end of the season, S(N) was more than 80% of N. Was there necessarily a moment in between when S(N) was exactly 80% of N?

(This is Problem A1 from the Sixty-fifth William Lowell Putnam Mathematical Competition given on Saturday, 4 December 2004. We congratulate Midshipmen James Brown, Andrew Bashelor, and Stephen McMath, who made up the Naval Academy Team in this competition.)

Each midshipman submitting a correct solution with a correct explanation to Problem 144 by 1700 Friday 17 December 2004 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his Math Dept mailbox in Preble Hall, second deck.

Correct solutions to Math Problem 143 were submitted by Midshipman Stephen S. McMath and Professors Mark Kidwell and Mark Meyerson. Partial solutions were submitted by Midshipmen Robert C. Gwinn, Daniel E. Harris, and Robert C. Miller.

A solution to Math Problem 143 is on the back of this page and is posted on the Math Dept bulletin board on the second floor of Preble Hall across from the stairs

MATHEMATICS PROBLEM 143

Show that the seven consecutive digits 2, 3, 4, 5, 6, 7, 8 can be arranged in 5040 different orders giving 5040 distinct seven digit integers, but none of these is a perfect square.

Solution

There are 7 choices for the first digit, and then 6 choices for the second digit, 5 for the third, 4 for the fourth, 3 for the fifth, 2 for the sixth, and only 1, the last remaining digit, for the seventh, so there are

$$7x6x5x4x3x2x1 = 7! = 5040$$

ways to arrange these seven digits. Since $10 = 1 \pmod 3$ it follows that $10^n = 1^n = 1 \pmod 3$ for any nonnegative integer n, and hence for any number b whose decimal digits sum to s,

$$b = b_n x 10^n + b_{n-1} x 10^{n-1} + \ldots + b_1 x 10^1 + b_0 10^0$$
$$= b_n + b_{n-1} + \ldots + b_1 + b_0 = s \pmod 3,$$

so b is congruent to $s \pmod 3$. That is, $b - s$ is an integer multiple of 3, or, equivalently, b and s have the same remainder when divided by 3. It follows that any one of the seven digit numbers satisfies

$$b = 2 + 3 + 4 + 5 + 6 + 7 + 8 = 35 = 2 \pmod 3.$$

Now $0^2 = 0$, $1^2 = 1$, and $2^2 = 1 \pmod 3$ are the only squares modulo 3, and so none of the seven digit numbers can be a perfect square.

Another way of looking at it: $(3k + 0)^2 = 9k^2 + 0 = 3h + 0$, $(3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3i + 1$, and $(3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 = 3j + 1$, so perfect squares can be a multiple of 3 or a multiple of 3 plus 1, but not a multiple of 3 plus 2. Since all of the 5040 numbers in question have remainder of 2 when divided by three, none is a perfect square.