

MATHEMATICS PROBLEM 146

A *palindrome* is a word (level, radar, eve), phrase or sentence (Able was I ere I saw Elba. A man, a plan, a canal, Panama.), or a number (21012, 6776, 81233218) which reads the same forward or backward.

How many prime numbers are palindromes with an even number of digits? State whether there are infinitely many or state the specific number of them. Give reasons which verify your answer.

Solution: There is exactly one prime number which is a palindrome with an even number of digits, namely, the prime number 11. This is true because every palindrome with an even number of digits is divisible by 11. To see this we observe that since $10 \equiv -1 \pmod{11}$ and hence $10^k \equiv (-1)^k \pmod{11}$, it follows that

$$\begin{aligned}n &= a_d a_{d-1} \dots a_1 a_0 = a_d 10^d + a_{d-1} 10^{d-1} + \dots + a_1 10 + a_0 \\ &\equiv a_d (-1)^d + a_{d-1} (-1)^{d-1} + \dots + a_1 (-1) + a_0 \\ &\equiv a_0 - a_1 + \dots \pm a_d \pmod{11}.\end{aligned}$$

That is, every number written in decimal notation is congruent modulo 11 to the alternating sum of its digits. But it is easy to see that the alternating sum of the digits of any palindrome with an even number of digits is 0, and hence any such number is divisible by 11.

Another approach: Those not acquainted with modular arithmetic might prefer the following approach. Let $u(n) = 11\dots 1$ be the n -digit number consisting of n ones, and let $v(n) = 100\dots 01$ be the n digit number consisting of two ones separated by $n-2$ zeros. Some small examples are $u(1) = 1$, $u(2) = v(2) = 11$, $u(3) = 111$, $v(3) = 101$, $u(4) = 1111$, $v(4) = 1001$, $u(8) = 11111111$, $v(8) = 10000001$. Clearly, $u(2n) = 11x10^{2n-2} + 11x10^{2n-4} + \dots + 11x10^2 + 11 = 11x1010\dots 101$ is divisible by 11 for each positive integer n , and hence $v(2n) = u(2n) - 10 \times u(2n-2)$ is also divisible by 11. Now $2332 = 2v(2) + 30v(1)$, $3141661413 = 3v(10) + 10v(8) + 400v(6) + 1000v(4) + 60000v(2)$, and similarly, every palindrome with an even number of digits is such a linear combination of $v(2k)$. As such, every palindrome with an even number of digits is divisible by 11.

MATHEMATICS PROBLEM 147

A spherical shaped experimental submarine has a spherical interior of radius 6 feet. The lower half of this sphere serves as a ballast tank. This tank has a flat top of pressure withstanding steel plate welded horizontally to the inside center of the sphere. On top of this plate is the "operating room" which is occupied by the sub's operator, air tanks, and equipment. The vessel is designed to float when the ballast tank is less than half full. (That is, when less than half of its volume is filled with water.) You are resting on the bottom of the Atlantic shelf at a depth of 200 feet when you discover that your ballast tank is full of water and your ballast blowing air tank is empty. How much work does it take to pump half of the sea water out of the ballast tank, assuming that sea water weighs 65 pounds per cubic foot and atmospheric pressure at the surface is 14.7 pounds per square inch?

Each midshipman submitting a correct solution with a correct explanation to Problem 147 by noon Thursday 31 March 2005 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or via his Math Dept mailbox in Preble Hall, second deck.

Correct solutions to Math Problem 146 were submitted by Midshipmen James P. Marek, Stephen S. McMath, Matthew D. Melton, and Daniel C. Ryan, as well as by Professor Mark D. Meyerson. Fifteen other midshipmen gave a correct answer without verifying that 11 divides every palindrome with an even number of digits.

A solution to Math Problem 146 is on the back of this page and is posted on the Math Dept bulletin board on the second floor of Preble Hall across from the stairs.