MATHEMATICS PROBLEM 149

A *monoid* is a set with a single binary operation (usually called multiplication, or addition) which is associative and which has an identity. For example, the set of all integers with multiplication as the binary operation and 1 as the identity, or the set of all 2×2 matrices with real entries with matrix multiplication as the binary operation and the 2×2 identity matrix I as the identity. If a is an element of a monoid M with binary operation $*$ and identity 1, then we define a^n inductively by taking $a^0 = 1$ and $a^n = (a^{n-1})*a$ for any positive integer n. Moreover, we say a is invertible with inverse b if and only if $a*b = 1 = b*a$.

Prove or give a counter example to each of the following statements for any a in a monoid M and any positive integer n:

1. If a is invertible, then so is a^n .
2. If a^n is invertible, then so is a .

Each midshipman submitting a correct solution with a correct explanation to Problem 149 by noon Friday 30 September 2005 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Math Problem 148 were submitted by Midshipman Stephen McMath and Professors Craig Bailey and Amy Ksir.

A solution to Math Problem 148 is on the back of this page and is posted on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 148

Suppose p and q are each monic polynomials of degree 4 with real coefficients and the intersection of their graphs is $\{(1, 3), (5, 21)\}$. If $p(3) - q(3) = 20$, what is the area enclosed by their graphs?

Solution. Let $r = p - q$ be the difference of the polynomials p and q. Then r is a polynomial of degree 3 or less satisfying $r(1) = 0$, $r(3) = 20$, and $r(5) = 0$. The area enclosed by the graphs of $y = p(x)$ and $y = q(x)$

is the integral of their difference $r(x)$ from $x = 1$ to $x = 5$. Since r is a third or lower degree polynomial, this integral is given exactly by Simpson's Rule, and so the area is

$$[r(1) + 4r(3) + r(5)](h/3) = [0 + (4)(20) + 0](2/3) = 160/3.$$

Remark. Could we have missed any area, or could the graphs have crossed somewhere in the middle, so that the integral includes a negative portion canceling part of the positive region? The answer is no to both of these questions. Since the graphs have exactly two points of intersection, the third or lower degree polynomial $r(x)$ has only two zeros, at 1 and at 3, so the graph of $y = r(x)$ is positive in the interval $(1, 5)$ and never 0 except at 1 and 5.

The next question is, do such polynomials exist? To show that they do, we construct the requisite p, q, and r, starting with r. Take $R(x) = (x-1)(x-5)$, so that $R(1) = R(5) = 0$, R has no other zeros, and $R(3) = -4$. Then $r_0(x) = -5R(x)$ is the unique quadratic which satisfies $r_0(1) = r_0(5) = 0$ and $r_0(3) = 20$. Moreover, for any constant c the polynomial

$r(x) = c(x-1)(x-3)(x-5)-5(x-1)(x-5)$ has degree 3 or less and satisfies $r(1) = r(5) = 0$ and $r(3) = 20$, and every polynomial of degree 3 or less which satisfies $r(1) = r(5) = 0$ and $r(3) = 20$ is of this form. Then $q(x) = x^4$ and $p(x) = q(x) + r(x)$ will do the job. Of course, there are infinitely many other choices for p and q which yield the same r; just let q be *any* monic fourth degree polynomial and take $p = q + r$.