# MATHEMATICS PROBLEM 150

Let $a$, $b$, and $c$ be real numbers and let $f$ and $g$ be real valued functions of a real variable such that $\lim_{x \to a} g(x) = b$ and $\lim_{x \to b} f(x) = c$.

a. Give an example in which $\lim_{x \to a} f(g(x)) \neq c$.

b. Give an additional condition on $f$ alone and show that it guarantees $\lim_{x \to a} f(g(x)) = c$.

c. Give an additional condition on $g$ alone and show that it guarantees $\lim_{x \to a} f(g(x)) = c$.

Each person submitting a correct solution with a correct explanation to Problem 150 by noon Friday 28 October 2005 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or by slipping under his office door, Chauvenet 375.

A very nice correct solution to Math Problem 149 was submitted by LT Josh Wood. It is posted on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

Another solution to Math Problem 149 is on the back of this page and is posted on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

## MATHEMATICS PROBLEM 149

A *monoid* is a set with a single binary operation (usually called multiplication, or addition) which is associative and which has an identity. For example, the set of all integers with multiplication as the binary operation and $1$ as the identity, or the set of all $2 \times 2$ matrices with real entries with matrix multiplication as the binary operation and the $2 \times 2$ identity matrix $I$ as the identity. If $a$ is an element of a monoid $M$ with binary operation $*$ and identity $1$, then we define $a^n$ inductively by taking

$a^0 = 1$ and $a^n = (a^{n-1})*a$ for any positive integer $n$. Moreover, we say $a$ is invertible with inverse $b$ if and only if $a*b = 1 = b*a$.

Prove or give a counter example to each of the following statements for any $a$ in a monoid $M$ and any positive integer $n$:

1. If $a$ is invertible, then so is $a^n$ .
2. If $a^n$ is invertible, then so is $a$ .

**Solution.** In order to simplify the notation, we indicate the binary operation * by juxtaposition, so that $x*y$ will be denoted by $xy$. We begin by using induction on $n$ to show that $a^n = aa^{n-1}$ for every positive integer $n$. The result holds for $n = 1$ because $a^1 = a^0 a = 1a = a = a1 = aa^0$. Suppose that the result holds for $n = k$; that is, $a^k = aa^{k-1}$. Then $a^{k+1} = a^k a = (aa^{k-1})a = a(a^{k-1}a) = aa^k$, and the result holds for $n = k + 1$. This completes the proof by induction that the result $a^n = aa^{n-1}$ holds for every positive integer $n$.

1. Suppose that $a$ is invertible with inverse $b$. We show by induction on $n$ that $a^n$ is invertible for every positive integer $n$. Clearly, the result holds for $n = 1$. Suppose that the result holds for $n = k$; that is, $a^k$ has inverse $c$, so that $a^k c = 1 = c a^k$ . Then $a^{k+1}(bc) = ( a^k a)(bc) = a^k(ab)c = a^k 1c = a^k c = 1$ and $(bc)a^{k+1} = (bc)(a^k a) = b(ca^k)a = b1a = ba = 1$. Thus, $a^n$ invertible for $n = k$ implies that $a^n$ is invertible for $n = k + 1$. This completes the inductive proof that $a^n$ is invertible for any positive integer $n$.

2. Suppose that $a^n$ is invertible with inverse $c$ for some positive integer $n$, so that $a^n c = 1 = ca^n$. Let $b = ca^{n-1}$. Then $ba = (ca^{n-1})a = c( a^{n-1}a) = ca^n = 1$. Let $d = a^{n-1}c$. Then $ad = a( a^{n-1}c) = (aa^{n-1})c = a^n c = 1$, since $aa^{n-1} = a^n$. But then $b = b1 = b(ad) = (ba)d = 1d = d$ is the (two sided) inverse of $a$.