MATHEMATICS PROBLEM 151

Consider the numbers 3, 8, 13, … , 103, 108; that is, all positive integers less than 110 which leave a remainder of 3 when divided by 5. What is the smallest k such that every collection of k of these numbers will always contain a pair which sums to 121?

Each person submitting a correct solution with a correct explanation to Problem 151 by noon Friday 25 November 2005 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!) or by slipping under his office door, Chauvenet 375.

A correct solution to Math Problem 150 was submitted by LT Josh Wood. Another solution to Math Problem 150 is on the back of this page and is posted on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 150

Let $a, b,$ and c be real numbers and let f and g be real valued functions of a real variable such that $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = c$.

a. Give an example in which $\lim_{x\to a} f(g(x)) \neq c$.

b. Give an additional condition on f alone and show that it guarantees $\lim_{x\to a} f(g(x)) = c$.

c. Give an additional condition on g alone and show that it guarantees $\lim_{x\to a} f(g(x)) = c$.

Solution. a. Let $a = b = c = 0,$ let $g(x) = 0$ for all x, and let $f(x) = 0$ for all $x \neq 0$ and $f(0) = 1.$ Then $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 0,$ but $\lim_{x\to 0} f(g(x)) = \lim_{x\to a} f(0) = \lim_{x\to a} 1 = 1 \neq 0.$

b. Let f be continuous at b. Then $\lim_{u\to b} f(u) = f(b) = c$. Hence, for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $0 < |u - b| < \delta_1$ implies $|f(u) - f(b)| < \varepsilon$. But $|f(u) - f(b)| = 0 < \varepsilon$ when $u = b$, so we see that $|u - b| < \delta_1$ implies $|f(u) - f(b)| < \varepsilon$. Since $\lim_{x\to a} g(x) = b$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|g(x) - b| < \delta_1$ implies (from above) that $|f(g(x)) - f(b)| < \varepsilon$. Therefore, $\lim_{x\to a} f(g(x)) = c$. □

c. Suppose there is a $d > 0$ such that $0 < |x - a| < d$ implies that $g(x) \neq b$. Then $\lim_{u\to b} f(u) = c$. Hence, for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that $0 < |u - b| < \delta_1$ implies $|f(u) - c| < \varepsilon$. Since $\lim_{x\to a} g(x) = b$, there is a $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $|g(x) - b| < \delta_1$. Let $\delta = \min(d, \delta_2)$. Then $0 < |x - a| < \delta$ implies $0 < |x - a| < d$ and $0 < |x - a| < \delta_2$ implies $g(x) \neq b$ and $|g(x) - b| < \delta_1$ implies $0 < |g(x) - b| < \delta_1$ implies $|f(g(x)) - c| < \varepsilon$. Therefore, $\lim_{x\to a} f(g(x)) = c$. □

Remark. Part b is just the standard composite function theorem for limits given in most elementary calculus books. For example, see James Stewart, *Calculus - Early Transcendentals,* 5th ed. Page 132 and pages A42-43. Part c is frequently used without mention in elementary calculus books when they state that $f'(a) = \lim_{x\to a} [f(x) - f(a)]/(x - a)$
$= \lim_{h\to 0} f[f(a+h) - f(a)]/h$.