# MATHEMATICS PROBLEM 154

Find the volume of the intersection of three cylinders, each of radius $a$, which are centered on the x-axis, the y-axis, and the z-axis. That is, find the volume of the three dimensional region

$$E = \{(x,y,z) : x^2 + y^2 \le a^2,\ y^2 + z^2 \le a^2,\ z^2 + x^2 \le a^2\}.$$

Each person submitting a correct solution with a correct explanation to Problem 154 by noon Friday 25 February 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Math Problem 153 were submitted by LT Josh Wood and by Professor Mark Meyerson. My solutions to Math Problem 153 are on the back of this page, and they, along with four solutions by Professor Meyerson, are posted on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

## MATHEMATICS PROBLEM 153

A set $S$ of points in space is *convex* if whenever two points are in $S$, then every point on the straight line segment joining these two points is also in the set $S$. If $S$ is any set of points in space, then the *convex hull* of $S$ is the smallest convex set containing $S$.

Find the volume of the convex hull of the set

$$S = \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}.$$

**Solution.** <u>Method 1 using facts from Calculus III:</u> The four given points are vertices of the unit cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. They determine three vectors $\underline{a} = \langle 1, 1, 0\rangle$,

which is a diagonal of the bottom $z = 0$ face of the cube, $\underline{b} = \langle 1, 0, 1\rangle$, which is a diagonal of the $y = 0$ face of the cube, and $\underline{c} = \langle 0, 1, 1\rangle$, which is a diagonal of the $x = 0$ face of the cube. These three vectors are adjacent edges of the tetrahedron $T$ that is the convex hull of $S$. The volume of the tetrahedron is $(1/3)Bh$, where $B$ is the area of its base and $h$ is its height. The triangular face with edges $\underline{a}$ and $\underline{b}$ has area $B = (1/2)|\underline{a} \times \underline{b}|$. Now $\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin(\theta)\, \underline{n}$, where $\theta$ is the angle between $\underline{a}$ and $\underline{b}$ and $\underline{n}$ is a unit vector perpendicular to $\underline{a}$ and $\underline{b}$. The height $h$ from the base $B$ to the vertex given by $\underline{c}$ is $h = |\underline{n} \cdot \underline{c}|$. The required volume is hence

$$V = (1/3)Bh = (1/3)(1/2)(|\underline{a} \times \underline{b}|)|\underline{n} \cdot \underline{c}| = (1/6)|\underline{a}|\,|\underline{b}| \sin(\theta)|\underline{n} \cdot \underline{c}|$$
$$= (1/6)|(\underline{a} \times \underline{b}) \cdot \underline{c}| = (1/6)|(\langle 1, 1, 0\rangle \times \langle 1, 0, 1\rangle) \cdot \langle 0, 1, 1\rangle| =$$
$$(1/6)\,|\langle 1, -1, -1\rangle \cdot \langle 0, 1, 1\rangle| = (1/6)|-2| = 1/3.$$

<u>Method 2 using facts from Calculus II:</u> The points $A = (0,0,0)$, $B = (0,1,1)$, $C = (1,0,1)$, and $D = (1,1,0)$ are the vertices of the tetrahedron which is their convex hull. This tetrahedron lies within the unit cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. The line segments $AD$ and $BC$ are diagonals of the bottom and top faces of the cube, respectively. The horizontal plane at height $z$ intersects the triangles $ADB$ and $ADC$ in line segments of length (°2)(1-z) parallel to $AD$, and it intersects the triangles $BCA$ and $BCD$ in line segments of length (°2)z parallel to $BC$. These four line segments form a rectangle of area $A(z) = 2z(1-z)$, the cross sectional area of the trapezoid perpendicular to the z-axis at height $z$. Hence the volume is

V = °(2*z*(1-z), z, 0, 1) = 1/3.

(The integral is given in Voyage 200 notation.)