MATHEMATICS PROBLEM 155

We can represent a triangle with sides of length *a, b, c* by the ordered triple *(a, b, c)*. Changing the order of the sides doesn't change the triangle, so *(a, b, c), (b, a, c), (b, c, a), (c, b, a), (c, a, b),* and *(a, c, b)* all represent the same triangle. To avoid confusion, let's agree to write *(a, b, c)* with $a \le b \le c$. We say that a triangle *(a, b, c)* is *integral* if *a, b,* and *c* are integers.

How many integral triangles are there with longest side less than or equal to *100* ?

Each person submitting a correct solution with a correct explanation to Problem 155 by noon Friday 24 March 2006 will be recognized as a solver on the next problem. Submit solutions to Prof. Wardlaw at mathprob@usna.edu (please no attachments!).

Correct solutions to Math Problem 153 were submitted by LT Josh Wood and by Professors Russell Jackson and Mark Meyerson. My solution to Math Problem 154 is on the back of this page and on the Math Dept bulletin board on the third floor of Chauvenet Hall across from the Mathematics Department Office.

MATHEMATICS PROBLEM 154

Find the volume of the intersection of three cylinders, each of radius $\boldsymbol{a}$, which are centered on the x-axis, the y-axis, and the z-axis. That is, find the volume of the three dimensional region

$$E = \{(x,y,z) : x^2 + y^2 \le \boldsymbol{a}^2,\ y^2 + z^2 \le \boldsymbol{a}^2,\ z^2 + x^2 \le \boldsymbol{a}^2\}.$$

Solution. We will first calculate the volume when $\boldsymbol{a} = 1$. By symmetry, the volume is sixteen times the volume in the half first octant given by $0 \le y \le x$, $x^2 + y^2 \le 1$, and $0 \le z \le (1 - x^2)^{1/2}$, which can be calculated using a triple integral in cylindrical coordinates. Hence the volume for $\boldsymbol{a} = 1$ is

When the radius of each cylinder is $\boldsymbol{a}$, we multiply by $\boldsymbol{a}^3$ to get the volume

Professor Jackson cleverly observed that each octant has a 3-fold symmetry (Rotations of 120° about the axis from the origin through the point (1, 1, 1) transform the solid into itself.), and each third of an octant has a reflection symmetry, so the total volume is 48 times the volume of any sixth octant, or

as obtained above.